The shape of probability judgments revisited

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September 1, 2015

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Abstract

The "two-stage" cumulative prospect theory (CPT) model offers an appealing explanation about the relationship between probability judgments, on the one hand, and CPT decision weights, on the other hand. Although monotonicity of probability judgments is required by this model, empirical studies to date do not control for monotonicity violating responses when they determine the typical shape of probability judgments. To fill this gap, we investigate in how far the inclusion of monotonicity violating responses in the data might result in a distorted picture about the shape of probability judgments. Our empirical findings show that the inclusion of monotonicity violating responses, firstly, strongly blows up the size of the concave and convex regions of inverse S-shaped probability judgments. Secondly, the dominance of the concave over the convex region, as observed under the exclusion of monotonicity violating responses, becomes reversed through inclusion.

Keywords: Cumulative Prospect Theory; Two-Stage Approach; Likelihood Insensitivity; Monotonicity Violations
JEL Classification Numbers: C91, D81

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1 Introduction

When we learn at school or university to think of chance in terms of empirical frequencies, we are taught an additive concept of probabilities; that is, probabilities of disjoint events have to sum up to the probability of the union of these events. In reality, however, most people are subject to cognitive biases such as, e.g., *likelihood insensitivity* (cf. Wakker 2010 and references therein), which lead to systematic violations of additivity in expressed probability judgments. Similarly, in choice situations under uncertainty most people systematically violate Savage's (1954) expected utility theory according to which their decision weights can be described as subjective additive probabilities. Descriptive decision theories like cumulative prospect theory (CPT) or Choquet expected utility theory therefore use non-additive probabilities as decision weights to express motivational factors such as, e.g., optimism versus pessimism or source-preference.[1] Although CPT could be used as a pure choice theory for which cognitive considerations are irrelevant[2], it is highly desirable to understand the relationship between the cognitive and the motivational realm. This holds in particular true for economists who want to use the rich survey data on probability judgments in CPT models; (cf. our discussion in Section 5 about life-cycle models and subjective survival beliefs).

The "two-stage" CPT model offers an elegant and intuitively appealing explanation about the relationship between probability judgments, on the one hand, and decision weights, on the other hand (see Wakker 2004 and references therein). According to this model, the decision maker forms in a first (=cognitive) stage beliefs (i.e., probability judgments) which are in a second (=motivational) stage transformed into CPT decision weights that express optimism versus pessimism attitudes for a given source of uncertainty. Formally, the "two-stage" CPT model can be expressed through the identity

$$W(E) = w(q(E)) \tag{1}$$

for all events E. Here, $W(\cdot)$ denotes a non-additive weighting function under uncertainty that determines the decision weight that a CPT decision maker attaches to any given event; $q(\cdot)$ denotes a non-additive function by which this decision maker judges the probability of any given event; and the residual $w(\cdot)$ stands for a weighting function under

[1] CPT, axiomatized in Tversky and Kahneman (1992) and in Wakker and Tversky (1993), combines non-additive decision weights with two different utility domains of outcomes separated by a reference points, i.e., *gains* versus *losses*. Restricted to gains CPT becomes formally equivalent to Choquet expected utility theory, axiomatized in Schmeidler (1989) and in Gilboa (1997), which uses the Choquet integral (cf. Schmeidler 1986) to integrate utilities with respect to a non-additive probability measure (i.e., decision weights under uncertainty).

[2] See, e.g., Abdellaoui, Vossmann, and Weber (2005).

risk which transforms judged probabilities into decision weights for the corresponding events.

The present paper analyzes the findings of an empirical study on the shape of probability judgments that are consistent with the "two-stage" CPT model. Our focus is thereby on possible distortions of this shape which might result from the inclusion of monotonicity violating responses in the data. By definition, CPT decision weights under uncertainty $W(\cdot)$ have to satisfy monotonicity (i.e., $E \subseteq E'$ implies $W(E) \subseteq W(E')$) in order to ensure the minimal rationality requirement of *stochastic dominance* in choice behavior. For the same reason, decision weights under risk $w(\cdot)$ have to be increasing (i.e., $q \leq q'$ implies $w(q) \leq w(q')$). As a trivial mathematical necessity, only such probability judgments $q(\cdot)$ that also satisfy monotonicity are therefore consistent with the identity (1).[3] Moreover, the violation of monotonicity has to be regarded as a much more severe cognitive shortcoming than mere likelihood insensitivity. The (open) question therefore arises whether such cognitive irrationality can still be associated with boundedly rational CPT maximization behavior or whether it rather co-occurs with downright irrational choice behavior such as violations of stochastic dominance. In the latter case, CPT would not be the adequate descriptive model because it excludes, by assumption, such irrationalities.

Existing empirical studies on the shape of probability judgments $q(\cdot)$ entering into the "two-stage" CPT model do not control for monotonicity violations. As a consequence, the existing empirical findings on the typical shape of q in the identity (1) might be distorted through the inclusion of monotonicity violating responses in the data. Recall that the existing literature agrees on the following stylized facts about the typical shape of probability judgments and decision weights, respectively.[4]

1. Both, CPT decision weights under uncertainty $W(\cdot)$ as well as probability judgments $q(\cdot)$, typically express an *inverse S-shape* characterized by a *concave region* for small events followed by a *convex region* for larger events.

[3] To see this suppose, to the contrary, that $q(\cdot)$ does not satisfy monotonicity; that is, there exist E and E' such that $E \subseteq E'$ and $q(E) > q(E')$. Because $w(\cdot)$ is increasing, (1) implies

$$W(E) > W(E'),$$

a contradiction to the presumed monotonicity of $W(\cdot)$. In this regard, the "two-stage" CPT model differs from the psychological "support theory" (e.g., Tversky and Koehler 1994), which allows for violations of monotonicity.

[4] This literature includes Tversky and Fox (1995), Fox, Rogers, and Tversky (1996), Fox and Tversky (1998), Wu and Gonzalez (1999), Kilka and Weber (2001).

2. Compared to probability judgments $q(\cdot)$, the inverse S-shape of decision weights $W(\cdot)$ tends to be more pronounced in the sense that the sizes of the concave and convex regions of $W(\cdot)$ tends to be greater than their counterparts for $q(\cdot)$.

3. Whereas the size of the convex region of decision weights $W(\cdot)$ strongly dominates the size of its concave region, the evidence is more mixed for probability judgments $q(\cdot)$ with a tendency towards a less pronounced dominance of the convex over the concave region.

We critically re-examine the above findings for probability judgments that are consistent with the "two-stage" CPT model in that they satisfy monotonicity. Our paper is divided into a theoretical and an empirical part. In the remainder of this introduction we discuss in more detail the respective contribution of each part.

Theoretical contribution

To describe the shape of probability judgments for which we have only a few data-points, we introduce several new concepts to the literature. First, we assume that we can formally associate any given probability judgment q with a corresponding function $f_q : [0, 2] \rightarrow [0, 2]$. An inverse S-shaped q is then characterized by a function f_q that has a unique fixed point $\alpha = f(\alpha)$ on the open interval $(0, 2)$ such that α is also the unique inflection point of q at which its concave region turns into its convex region. To associate each inverse S-shaped probability judgment with a unique inflection point provides us with strictly more information on the shape of probability judgments than the existing concepts of *bounded lower* and *upper subadditivity*.

In a next step, we stipulate that the data on probability judgments is generated by a representative agent who chooses inflection points in accordance with some random choice rule. That is, for any given α the representative agent makes an α-typical (described as empirical median) probability judgment whereby α is independently drawn from some fixed distribution over possible inflection point values α in $(0, 2)$. We then define the global measures for the 'size' of the concave, resp. convex, regions as the expected value over the respective sizes of all α-typical probability judgments with respect to the empirical distribution over the α values. We argue that these new global measures for the representative sizes of the concave and convex regions of inverse S-shaped probability judgments avoid the inconsistencies that arise for the existing Tversky and Fox (1995) measures whenever inflection points are not close to one.

Empirical contribution

We conducted an empirical study with 203 undergraduate commerce students from the University of Pretoria as participants who had to answer a questionnaire which asked for probability judgments for seven different sources of uncertainty. In total, we collected responses that correspond to 1357 different probability judgments whereby this data consisted of exactly three data-points per probability judgment. Based on these three data-points, we use restricted regression methods to calculate the unique inflection point of any inverse S-shaped probability judgment associated with the data.

The resulting empirical distribution of these inflection points is then used in our analysis of the shape of probability judgments, in particular, the calculation of the global measures for concave and convex regions. With respect to the features of the shape of inverse S-shaped probability judgments, we compare three different data scenarios: (i) only monotonicity consistent responses are included; (ii) only monotonicity violating responses are included; (iii) both types of responses are included. Whereas only the first scenario is consistent with the "two-stage" CPT model, the existing data on inverse S-shaped probability judgments would have been obtained from situations similar to the third scenario.[5]

Two novel empirical findings emerge. First, the inclusion of monotonicity violating responses strongly blows up the sizes of the concave and convex regions. Second, this inclusion also shifts the predominance from the concave to the convex region. In other words, the inclusion of monotonicity violating responses in the data indeed results in a distorted picture of the probability judgments of a representative "two-stage" CPT decision maker. These findings suggest that economic models which use the "two-stage" CPT model will need a much less linear decision weighting function under risk $w(\cdot)$ than has been previously thought.[6] That is, the function $w(\cdot)$ that transforms probability judgments into decision weights has to emphasize much more the inverse S-shape as well as the predominance of the convex over the concave region than suggested by previous studies.

Our paper is organized as follows. Section 2 introduces new theoretical concepts for the analysis of the shape of probability judgments of a representative agent, which

[5] We say "similar" because our approach restricts attention to responses that are consistent with an inverse S-shape, whereas existing studies apparently do not impose such restrictions. However, since the number of inverse S-shape violating responses in our study turns out to be negligible, there should be no relevant discrepancies between our third scenario and existing studies.

[6] For example, Fox, Rogers, and Tversky (1996) have suggested that $w(\cdot)$ is linear (i.e., the identity) for option traders who are supposedly almost risk-neutral.

we compare with existing concepts in Section 3. In Section 4 we describe the set-up of our empirical study while Section 5 presents our findings. Section 6 discusses the implications of our findings for economic modelling. Section 6 concludes.

2 New theoretical concepts

2.1 Probability judgments with an inverse S-shape

Denote by S a *state space* and by Σ some *sigma-algebra* defined on S. That is, Σ consists of subsets of S, including the empty set $\emptyset$ and S itself, and it is closed under countable intersections and unions of its members. The members of S are called *events*. Fix the measurable space (S, Σ). A *probability judgment* on (S, Σ), denoted q, is any mapping from the event space Σ into the unit interval $[0, 1]$ that satisfies *normality*, i.e.,

$$\begin{aligned} q(\emptyset) &= 0, \\ q(S) &= 1. \end{aligned}$$

The set of all probability judgments on (S, Σ) is denoted $\mathbb{Q}$. To describe the local shape of probability judgments, we use the following standard definitions for fixed events $E, E' \in \Sigma$.

Definitions "Convexity, Concavity, and Additivity".

(i) $q \in \mathbb{Q}$ satisfies (*strict*) *concavity at* $E, E' \in \Sigma$ if, and only if,

$$q(E \cup E') + q(E \cap E') \leq (<)\, q(E) + q(E'). \tag{2}$$

(ii) $q \in \mathbb{Q}$ satisfies (*strict*) *convexity at* $E, E' \in \Sigma$ if, and only if,

$$q(E \cup E') + q(E \cap E') \geq (>)\, q(E) + q(E'). \tag{3}$$

(iii) $q \in \mathbb{Q}$ satisfies *additivity at* $E, E' \in \Sigma$ if, and only if, it satisfies convexity as well as concavity at $E, E' \in \Sigma$, i.e.,

$$q(E \cup E') + q(E \cap E') = q(E) + q(E').$$

The global shape of a probability judgment can be highly complex, because it might be strictly concave at some events while being strictly convex at other events. The following definitions identify perceivable simple structures for the global shape of probability judgments.

Definitions "Concavity below α, convexity above α".

Fix some $\alpha \in (0,2)$.

(i) $q \in \mathbb{Q}$ satisfies *concavity below* α if, and only if, for all $E, E' \in \Sigma$,

$$q(E \cup E') + q(E \cap E') \leq \alpha \tag{4}$$

implies

$$q(E \cup E') + q(E \cap E') \leq q(E) + q(E').$$

(ii) $q \in \mathbb{Q}$ satisfies *convexity above* α if, and only if, for all $E, E' \in \Sigma$,

$$q(E \cup E') + q(E \cap E') \geq \alpha \tag{5}$$

implies

$$q(E \cup E') + q(E \cap E') \geq q(E) + q(E').$$

For analytical convenience, we assume that probability judgments satisfy the following structural assumptions.

Assumptions. Fix $q \in \mathbb{Q}$ and define, for all $E, E' \in \Sigma$,

$$\begin{aligned} x_{E,E'} &\equiv q(E \cup E') + q(E \cap E'), \\ f_{E,E'} &\equiv q(E) + q(E'). \end{aligned}$$

A1. We assume that, for all $E, E', E'', E''' \in \Sigma$,

$$x_{E,E'} = x_{E'',E'''} \equiv x \text{ implies } f_{E,E'} = f_{E'',E'''} \equiv f_q(x).$$

A2. We further assume that the structure of the event space Σ is sufficiently *fine* in the sense that there exists for every $x \in [0,1]$ some disjoint $E, E' \in \Sigma$ such that $x = x_{E,E'}$.

By Assumption 1, the global shape of any given $q \in \mathbb{Q}$ can be equivalently described by the corresponding function $f_q : [0,2] \to [0,2]$ such that f_q must have the following two fixed points: $f_q(0) = 0$ and $f_q(2) = 2$.[7] We also have the following formal relationships between the shape of q and properties of f_q.

Observation 1.

(i) $q \in \mathbb{Q}$ satisfies *concavity below* α if, and only if, for all $x \in (0, \alpha]$,

$$f_q(x) \geq x. \tag{6}$$

(ii) $q \in \mathbb{Q}$ satisfies *convexity above* α if, and only if, for all $x \in [\alpha, 2)$,

$$f_q(x) \leq x. \tag{7}$$

(iii) If $q \in \mathbb{Q}$ satisfies *concavity below* α as well as *convexity above* α, there exists a fixed point at α, i.e.,

$$f_q(\alpha) = \alpha. \tag{8}$$

Definition "Inverse S-shape". We say that $q \in \mathbb{Q}$ exhibits an *inverse S-shape* if, and only if, q satisfies *concavity below* α as well as *convexity above* α for a unique $\alpha \in (0, 2)$. We denote the set of all probability judgments exhibiting an inverse S-shape as $\mathbb{Q}^{invS}$.

If q exhibits an inverse S-shape, it has, by definition, a unique inflection point, given by α, at which the shape of q turns from concave to convex. Note that the inflection point α of q can be equivalently characterized as a fixed point of f_q. More precisely, we obtain the following characterization of probability judgments q that exhibit an inverse S-shape in terms of the corresponding functions f_q.

[7] To see this, note that

$$\begin{aligned} x_{\emptyset,\emptyset} &= f_{\emptyset,\emptyset} = 0, \\ x_{S,S} &= f_{S,S} = 2. \end{aligned}$$

Observation 2. $q \in \mathbb{Q}^{invS}$ if, and only if, for some $\alpha \in (0,2)$,

$$\begin{aligned} f_q(0) &= 0, \\ f_q(x) &> x \text{ for all } x \in (0,\alpha), \\ f_q(\alpha) &= \alpha, \\ f_q(x) &< x \text{ for all } x \in (\alpha,2), \\ f_q(2) &= 2. \end{aligned}$$

2.2 New measures for convex and concave regions

We suppose that the data on empirical probability judgments can be thought of as the probability judgments of a representative agent who uses a fixed random choice rule. More specifically, the representative agent applies a random choice rule according to which the inflection points α of his inverse-S shaped probability judgments are independently drawn from a fixed probability distribution corresponding to an objective probability measure, denoted π, defined over the Borel sets on $[0,2]$. For each drawn α-value, the representative agent then chooses an α-typical judgment, which we estimate from the data as the probability judgment that has the median 'sizes' of the concave and convex regions over all probability judgments in the data that have inflection point α. In what follows, we elaborate on this novel approach.

Suppose that the representative agent's probability judgments which exhibit an inverse S-shape are empirically given by the finite data-set $\mathbb{Q}^* \subset \mathbb{Q}^{invS}$. In the empirical part of the paper, we propose a method by which we estimate for every probability judgment its unique inflection point resulting in a finite set

$$\mathcal{A} = \{0.01, 0.02, ..., 1.98, 1.99\} \subset (0,2)$$

of estimated α values. Denote by $q_\alpha \in \mathbb{Q}^*$ a probability judgment with inflection point $\alpha \in \mathcal{A}$, and consider the following empirical frequency of probability judgments with inflection point α

$$\tilde{\pi}(\alpha) = \frac{\#\{q_\alpha \in \mathbb{Q}^*\}}{\#\mathbb{Q}^*}.$$

We will use the empirical probability measure $\tilde{\pi}$ over the estimated values of α as an estimate for the objective probability measure π.

Denote by $\mathbb{E}[\alpha]$ the empirical mean over all admissible inflection points of the probability judgments in $\mathbb{Q}^*$, i.e.,

$$\mathbb{E}[\alpha] = \sum_{\alpha \in \mathcal{A}} \alpha \cdot \tilde{\pi}(\alpha).$$

This expected value of inflection points and their empirical median provide us already with valuable information about the shape of probability judgments and we will use it in our data-analysis.

To obtain a picture about the pronouncement of the inverse S-shape from a few data-points, we need empirical measures for the typical 'sizes' of concave and concave regions of inverse S-shaped probability judgments. To this purpose, consider a finite collection $\mathbb{C}$ of event-pairs (E, E'). Define, for a given probability judgment $q_\alpha \in \mathbb{Q}^*$ with inflection point α, the two sets

$$\begin{aligned} X_{q_\alpha}^{cave} &= \{x \in [0, \alpha) \mid x \equiv q_\alpha (E \cup E') + q_\alpha (E \cap E') < \alpha, (E, E') \in \mathbb{C}\}, \\ X_{q_\alpha}^{vex} &= \{x \in (\alpha, 2] \mid x \equiv q_\alpha (E \cup E') + q_\alpha (E \cap E') > \alpha, (E, E') \in \mathbb{C}\}. \end{aligned}$$

The interpretation is that $X_{q_\alpha}^{cave}$ collects the data points x that lie in the strictly concave region of the probability judgment q_α whereas $X_{q_\alpha}^{vex}$ collects the data points x that lie in the strictly convex region of q_α. Next, define for every $q_\alpha \in \mathbb{Q}^*$ the average of the distances between $f_{q_\alpha}(x)$ and x (resp. between x and $f_{q_\alpha}(x)$) as a measure of the 'size' of the concave (resp. convex) region of q_α:

$$\begin{aligned} d^{cave}(q_\alpha) &\equiv \frac{1}{\# X_{q_\alpha}^{cave}} \sum_{x \in X_{q_\alpha}} f_{q_\alpha}(x) - x, \\ d^{vex}(q_\alpha) &\equiv \frac{1}{\# X_{q_\alpha}^{vex}} \sum_{x \in X'_\alpha} x - f_{q_\alpha}(x). \end{aligned}$$

Note that there might be different probability judgments q_α, q'_α in $\mathbb{Q}^*$ that share the same inflection point α but give rise to different sizes, i.e., $d^{cave}(q_\alpha) \neq d^{cave}(q'_\alpha)$ or $d^{vex}(q_\alpha) \neq d^{vex}(q'_\alpha)$. To determine the sizes of the α-typical probability judgment, we consider with $\mathbb{M}_\alpha^{cave}$, resp. $\mathbb{M}_\alpha^{vex}$, the empirical median of all sizes $d^{cave}(q_\alpha)$, resp. $d^{vex}(q_\alpha)$, $q_\alpha \in \mathbb{Q}^*$ for any fixed value $\alpha \in \mathcal{A}$.

Definition "Empirical Measures for Concave and Convex Regions of the Representative Inverse S-shaped Probability Judgment".

(i) The empirical measure for the (*representative*) *concave region* is defined as the expected value, w.r.t. $\tilde{\pi}$, of the concave sizes of all α-typical probability judgments, i.e.,

$$\mathbb{E}[\mathbb{M}_\alpha^{cave}] = \sum_{\alpha \in \mathcal{A}} \mathbb{M}_\alpha^{cave} \cdot \tilde{\pi}(\alpha).$$

(ii) The global measure for the (*representative*) *convex region* is defined as the expected value, w.r.t. $\tilde{\pi}$, of the convex sizes of all α-typical probability judgments, i.e.,

$$\mathbb{E}\left[\mathbb{M}_{\alpha}^{vex}\right] = \sum_{\alpha\in\mathcal{A}} \mathbb{M}_{\alpha}^{vex} \cdot \tilde{\pi}\left(\alpha\right).$$

Recall that a probability judgment q satisfies *monotonicity* if, and only if, for all $E, E' \in \Sigma$,

$$E \subset E' \text{ implies } q\left(E\right) \leq q\left(E'\right). \tag{9}$$

Further note that monotonicity of q is equivalent to a weakly increasing function f_q so that violations of monotonicity imply that f_q must be strictly decreasing for some arguments x. As a consequence, the size of the concave (resp. convex) region of a weak monotonicity violating q_α tends to be greater than the corresponding size of some q'_α that satisfies weak monotonicity. While a rigorous mathematical formalization of this statement is beyond the scope of the present paper, the intuition behind the argument can be easily grasped from Figure 1.

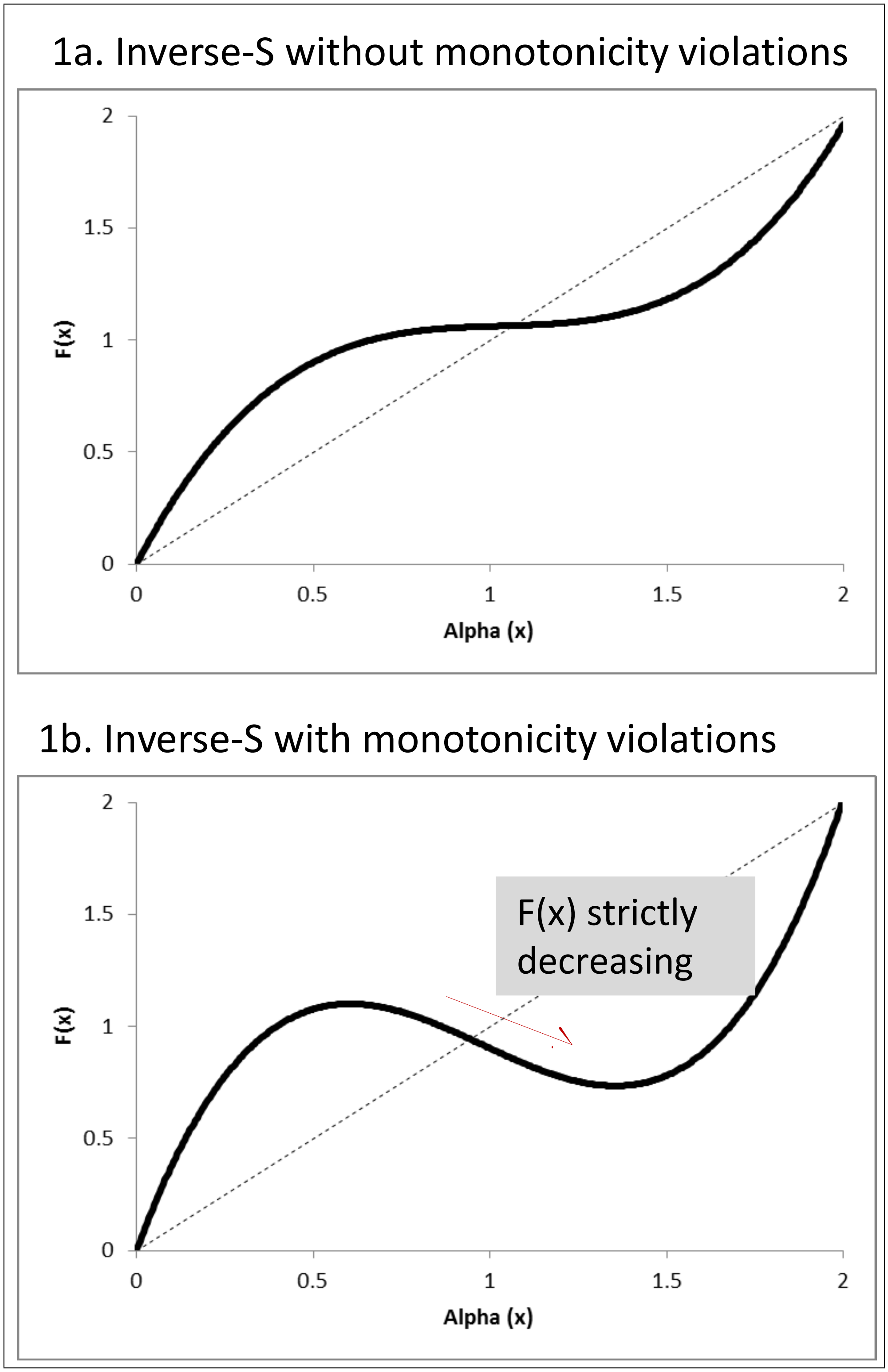

Figure 1: Inverse-S function with and without violations of monotonicity

3 Comparison with the existing literature

This section compares our new theoretical concepts from the previous section with the corresponding standard concepts used in the existing literature. (The reader who is foremostly interested into our empirical findings might want to skip this theoretical section.)

3.1 Subadditivity

The standard concepts for describing inverse S-shaped probability judgments in the existing literature are *unbounded lower* and *upper subadditivity* (SA).

Definition "Bounded Subadditivity".

(i) $q \in \mathbb{Q}$ satisfies ε-*bounded lower SA* if, and only if, there is some $\varepsilon \geq 0$ such that, for all disjoint events $D, D' \in \Sigma$ with $q(D \cup D') \leq 1 - \varepsilon$,

$$q(D \cup D') \leq q(D) + q(D'). \tag{10}$$

(ii) $q \in \mathbb{Q}$ satisfies ε'-*bounded upper SA* if, and only if, there is some $\varepsilon' \geq 0$ such that, for all disjoint events $D, D' \in \Sigma$ with $q(S - D \cup D') \geq \varepsilon'$,

$$q(S) - q(S - D') \geq q(S - D) - q(S - D \cup D'). \tag{11}$$

Lower SA is often interpreted as a possibility effect: Adding an uncertain event D' to another uncertain event D has less impact on the increase in judged probability than just adding event D' to the impossible null/empty event (i.e., introducing D as a possibility). Conversely, upper SA has been interpreted as a certainty effect: Taking an uncertain event D' away from the certain event S (i.e., turning certainty into a mere possibility) has greater impact on the decrease of judged probabilities than just taking D' away from another uncertain event $S - D$ (i.e., turning possibility into a less likely possibility).

Both SA concepts are closely related to our definitions of concavity below α, convexity above α, and an inverse S-shape. In particular, we prove in Appendix A that (i) ε-bounded lower SA is equivalent to concavity below $1 - \varepsilon$ whenever we restrict attention to disjoint events $E = D$ and $E' = D'$ whereas (ii) ε'-bounded upper SA is equivalent to convexity above $1 + \varepsilon'$ whenever we restrict attention to events $E = S - D$ and $E' = S - D'$ such that D and D' are disjoint.

Observation 3.

(i) If $q \in \mathbb{Q}$ satisfies *concavity below* $\alpha = 1 - \varepsilon$, then q also satisfies ε-*bounded lower subadditivity.*

(ii) If $q \in \mathbb{Q}$ satisfies *convexity above* $\alpha = 1 + \varepsilon'$, then q also satisfies ε'-*bounded upper subadditivity.*

(iii) Suppose that q exhibits an inverse S-shape, i.e., $q \in \mathbb{Q}^{invS}$. Then q also satisfies ε-*bounded lower* as well as ε'-*bounded upper subadditivity* if, and only if, it holds for the unique inflection point that $\alpha \in [1 - \varepsilon, 1 + \varepsilon']$.

Whenever probability judgments exhibit an inverse S-shape, the special case of *unbounded* SA, i.e., $\varepsilon = \varepsilon' = 0$, requires that all these probability judgments have an unique inflection point at $\alpha = 1$. But to restrict attention to probability judgments with a unique inflection point value of one is highly restrictive. In contrast, the more general case of bounded SA with $\varepsilon, \varepsilon' > 0$ is consistent with all inverse S-shaped probability judgments that have their inflection point value in the interval $[1 - \varepsilon, 1 + \varepsilon']$. While greater boundaries $\varepsilon, \varepsilon' > 0$ thus allow us to included more inverse S-shaped probability judgments into the analysis, greater boundaries also have the drawback that we lose more information about inverse S-shaped probability judgments that satisfy bounded SA: Instead of knowing for each probability judgment its unique inflection point value α, we only know that this value lies in the interval $[1 - \varepsilon, 1 + \varepsilon']$.

To summarize: Whereas our definition of an inverse S-shape associates any inverse S-shaped probability judgment with a unique inflection point value, the concepts of ε-bounded lower and ε'-bounded upper subadditivity associate any inverse S-shaped probability judgment with the interval $[1 - \varepsilon, 1 + \varepsilon']$ of possible inflection point values. As one consequence, these existing concepts cannot make use of our assumption that inflection points are i.i.d. when deriving the representative sizes of concave and convex regions of inverse S-shaped probability judgments from the data.

3.2 Tversky-Fox measures for subadditive regions

Because the important information about the unique inflection point of any given inverse S-shaped probability judgment is not lost under our global measures for the concave, resp. convex, regions, our new definitions are able to avoid severe inconsistencies of the existing definitions. To see this, let us recall these existing concepts for global measures of lower, resp. upper, SA, whereby our formalism closely follows the verbal description in Tversky and Fox (1995).

Consider a collection $\mathbb{C}'$ of pairs of non-empty disjoint events (D, D') such that $D \cup D' \subset S$. Define for a given inverse S-shaped probability judgment $q \in \mathbb{Q}^*$ the two sets

$$\begin{aligned} X_q^{LSA} &= \{q(D \cup D') \mid (D, D') \in \mathbb{C}'\}, \\ X_q^{USA} &= \{q(S) + q(S - D \cup D') \mid (D, D') \in \mathbb{C}'\}. \end{aligned}$$

Here, X_q^{LSA} collects the data points x for which the 'size' of lower SA is measured for q whereas X_q^{USA} collects the data points x for which the 'size' of upper SA is measured for q. Next, define for every $q \in \mathbb{Q}^*$ the average of the distances between $f_q(x)$ and x (resp. between x and $f(x)$):

$$\begin{aligned} d^{LSA}(q) &\equiv \frac{1}{\#X_q^{LSA}} \sum_{x \in X_q^{LSA}} f_q(x) - x, \\ d^{USA}(q) &\equiv \frac{1}{\#X_q^{USA}} \sum_{x \in X_q^{USA}} x - f_q(x). \end{aligned}$$

Definition "Tversky-Fox Measures for Subadditivity". Given some data-set $\mathbb{Q}^* \subset \mathbb{Q}^{invS}$.

(i) The Tversky-Fox measure for lower SA, denoted $\mathbb{M}^{LSA}$, is defined as the empirical median of all sizes $d^{LSA}(q)$ with $q \in \mathbb{Q}^*$.

(ii) The Tversky-Fox measure for upper SA, denoted $\mathbb{M}^{USA}$, is defined as the empirical median of all sizes $d^{USA}(q)$ with $q \in \mathbb{Q}^*$.

The Tversky-Fox measures may give us a distorted picture of the size of the concave, resp. convex, regions if the inflection point α of the probability judgment in question is not close to one. Indeed, the Tversky-Fox measures might even take on negative values in case α is close to zero or two. The mathematical reason for this distortion is that the calculations of $d^{LSA}(q)$ and $d^{USA}(q)$ tend to mix up convex with concave regions for probability judgments whose inflection points do not coincide with one. This can be seen in Figure 2: the Tversky-Fox approach would assume that both 1. and 2. in the figure below fall in the convex region, since both are below $f(x) = x = 1$. In fact, 2. falls in the convex region in this example.

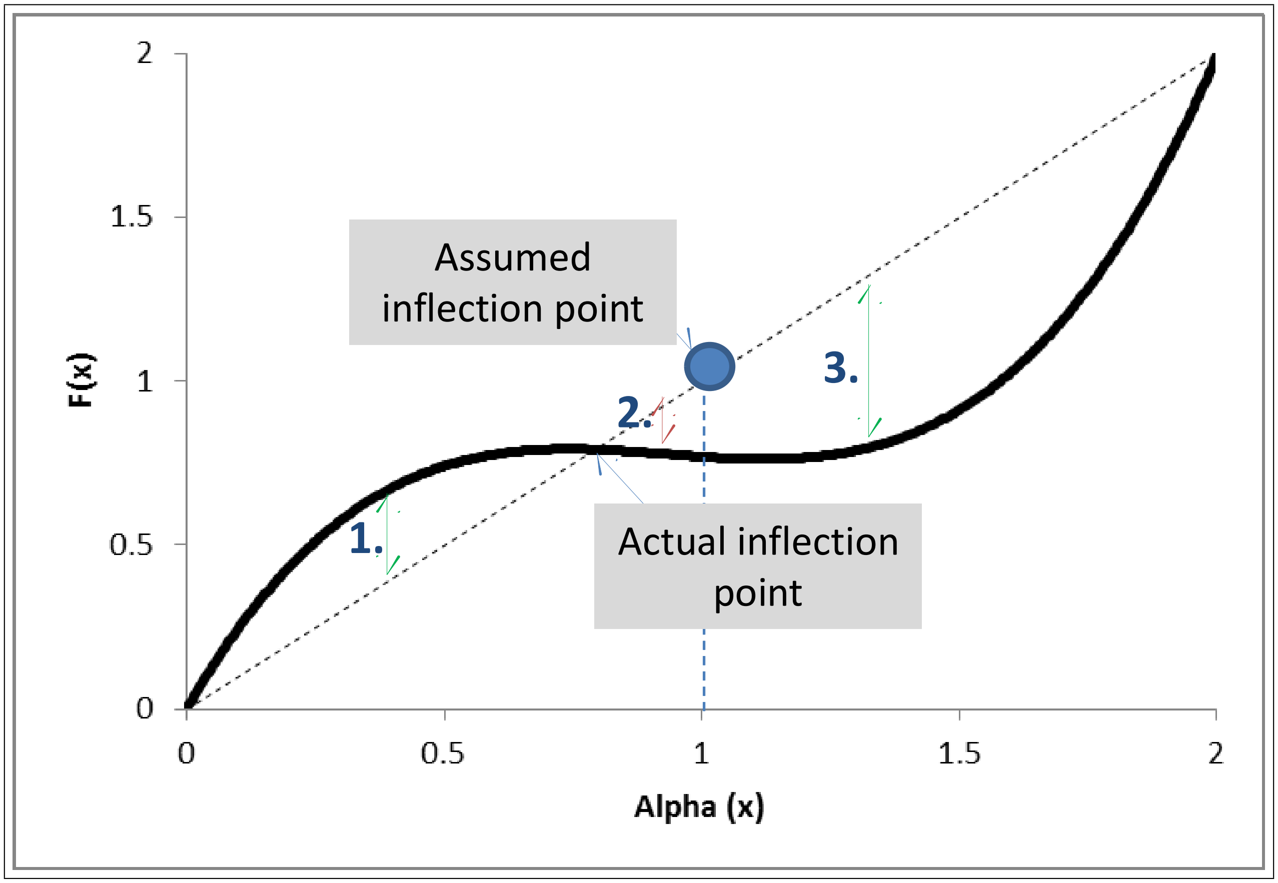

Figure 2: Impact of alpha values different from one

In other words, the Tversky-Fox measures perform well as long as only probability measures with an inflection point close to one are considered. But then the Tversky-Fox measures become a degenerate special case of our new global measures, which can be seen from the following observation.

Observation 4. Restrict attention to a finite collection $\mathbb{C}$ of event-pairs (E, E') such that either $E = D$ and $E' = D'$ or $E = S - D$ and $E' = S - D'$ for disjoint D and D'. In general, our new global measures coincide on $\mathbb{Q}^*$ with the above existing measures, i.e.,

$$\mathbb{E}\left[\mathbb{M}_{\alpha}^{cave}\right] = \mathbb{M}^{LSA} \text{ and } \mathbb{E}\left[\mathbb{M}_{\alpha}^{vex}\right] = \mathbb{M}^{USA} ,$$

if, and only if, the empirical distribution over inflection points reduces to the degenerate measure that attaches probability one to the inflection point value of one, i.e.,

$$\tilde{\pi}(\alpha = 1) = 1.$$

Proof: tbc. if-part: $X_{q_\alpha}^{cave} = X_{q_\alpha}^{LSA}$ and $X_{q_\alpha}^{vex} = X_{q_\alpha}^{USA}$

4 Empirical strategy

4.1 Study design

We recruited 203 undergraduate commerce students at the University of Pretoria. Students were asked to stay after class to complete a paper questionnaire. In order not to bias results in any way, students were simply told that we were interested in their responses for a study on decision making. In line with the ethics policy of the university, students were guarenteed anonymity of responses. The questionnaire first presented an example probability judgment: the likelihood of rain on the university campus on a given future date. Details were provided for this example to illustrate how likelihood judgments should be reported: "If you're absolutely certain that it will rain on this specific date in this location, you would answer 100%, while if you are absolutely certain it will not rain, you would answer 0%. If you believe there is a 63% chance of rain, you would give this percentage as your response."

Following this example, our questionnaire asked questions about probability judgments for seven different sources of uncertainty (all defined on the same state space S):

Source I: Inflation.

Source II: Weather.

Source III: Balls in a Bag.

Source IV: Rugby.

Source V: The Stock Exchange.

Source VI: The Rand/US Dollar Exchange Rate.

Source VII: The Thai Baht/US Dollar Exchange Rate.

We associate with each source $k \in \{I, ..., VII\}$ exactly five different events in Σ, denoted $A1_k$, $A2_k$, $B1_k$, $B2_k$, and C_k. For each source, these events are constructed such that they give rise to the following three partitions of the state space S, whereby we drop the subscript k from now on:

$$\{A1, A2\}, \quad (12)$$

$$\{B1, B2\}, \quad (13)$$

$$\{A1, C, B2\}. \quad (14)$$

We further impose, by construction, the following strict set-inclusions

$$A1 \subset B1, \tag{15}$$

$$B2 \subset A2 \tag{16}$$

as well as the intersection

$$C = A2 \cap B1 \tag{17}$$

implying

$$C \subset A2, B1. \tag{18}$$

Consequently, a probability judgment q is consistent with weak monotonicity if, and only if, it satisfies the following inequalities:

$$q(A1) \leq q(B1), \tag{19}$$

$$q(B2) \leq q(A2), \tag{20}$$

$$q(C) \leq q(A2), q(B1). \tag{21}$$

As an illustrative example consider Source I: Inflation. Without specifying the state space S in detail, observe that the according events $A1$, $A2$, $B1$, $B2$, and C are in one-one correspondence with subsets of the positive real-line so that we simply write:

$$\begin{aligned} A1 &= [0,3] \\ A2 &= (3,\infty) \\ B1 &= [0,6] \\ B2 &= (6,\infty) \\ C &= (3,6]. \end{aligned}$$

4.2 Calculation of inflection points

Recall that we associate with each probability judgment q the function $f_q : [0,2] \rightarrow [0,2]$ such that

$$f_q(x) = q(E) + q(E')$$

whereby the arguments $x \in [0,2]$ are defined as

$$x = q(E \cup E') + q(E \cap E')$$

for all $E, E' \in \Sigma$. Our study design elicits as responses three data-points $(x, f_q(x))$ which supposedly reflect values and arguments of some underlying probability judgment q. The possible values of the arguments are thereby given as follows:

$$x \in X_q \equiv \{q(A1 \cup C), q(B2 \cup C), q(A2 \cup B1) + q(A2 \cap B1)\}.$$

Observation 5. Suppose that $X_q = \{x_1, x_2, x_3\}$ such that $x_1 \leq x_2 \leq x_3$. The elicited data is NOT consistent with any probability judgment q exhibiting an inverse S-shape if, and only if, f_q satisfies any of the following two conditions:

1. $f(x_1) \leq x_1$ for $x_1 > 0$, and $f(x_2) \geq x_2$;
2. $f(x_2) \leq x_2$, and $f(x_3) \geq x_3$ for $x_3 < 2$.

The above conditions include, e.g., additivity of q at at least two different arguments x as well as convexity followed by concavity. Whenever the elicited data does NOT meet the excluding conditions in Observation 5, we assume that this data actually belongs to a unique probability judgment q exhibiting an inverse S-shape. To determine the unique inflection point α of any such q from three data-points only, we perform a linear regression across the three different elicited $x \in X_q$. The concave region has $x < f(x)$ while the convex region has $x > f(x)$. In order to find α, we therefore find the point on the linear regression through $x \in X_q$ where $x = f(x)$: the shift from concavity to convexity.

In an unrestricted linear regression, the α predicted from $x = f(x)$ is at times identified as a point where $x_\alpha > x_2$ or $x_\alpha > x_3$ where $x_2 > f(x_2)$ or $x_3 > f(x_3)$. That is, x_α is predicted to fall above a point known to fall in the upper sub-additive region. We therefore restrict the value of x_α such that this cannot be greater than the known $f(x_2)$ or $f(x_3)$ if $x_2 > f(x_2)$ or $x_3 > f(x_3)$. In other words, the inflection point must be below the upper sub-additive region.

Two feasible probability judgment structures require slightly different treatment:

1. In the case below, all elicited $x \in X_q$ fall in the upper sub-additive region. Here we assume that x_α must fall below $f(x_1)$. We thus calculate x_α for this case as the mid-point between 0 and $f(x_1)$.

$$f(x_1) > x_1, f(x_2) > x_2, f(x_3) > x_3;$$

2. In the case below, all elicited $x \in X_q$ fall in the lower sub-additive region. Here we assume that x_α must fall above $f(x_3)$. We thus calculate x_α for this case as the mid-point between 2 and $f(x_3)$.

$$f(x_1) < x_1, f(x_2) < x_2, f(x_3) < x_3.$$

Finally notice that we round all obtained estimates for inflection points to the closest value in the set

$$\mathcal{A} = \{0.01, 0.02, ..0.99, 1, 1.01, ..., 1.98, 1.99\}.$$

5 Empirical findings

5.1 Inverse S-shape and monotonicity violations

At first, we would like to establish the extent to which violations of monotonicity exist. Table 1 below summarizes the findings across sources for all respondents. While nearly three quarters of responses are consistent with inverse S- shaped probability judgments, only 22% of these are consistent with weak monotonicity, whereas 78% of these violate weak monotonicity. The predominance of responses consistent with an inverse S-shape–in violation of additivity–is even more impressive, if Source III: Balls in a bag was excluded (see the Remark below). Within responses consistent with monotonicity, 39% (218/559) demonstrate the inverse-S shape. This proportion becomes a (marginal) majority of 50.3% (201/399) where we exclude Source III: Balls in a bag.

Table 1: Additivity, Inverse S-shape, Monotonicity violations

	Total (%) (n=1357)	Balls (n=198)	Excl. Balls (n=1159)
Additive at all three points	193 (14%)	129 (65%)	64 (6%)
Additive at two points	42 (3%)	3 (2%)	39 (3%)
Consistent with monotonicity	559 (41%)	160 (81%)	399 (34%)
Consistent with inverse-S	987 (73%)	51 (26%)	936 (81%)
Inverse-S and monotonicity	218 (16%)	17 (9%)	201 (17%)

Whereas inverse S-shaped probability judgments can be plausibly explained by the cognitive shortcoming of *likelihood insensitivity*, violations of monotonicity stand for a much stronger irrationality in cognitive judgments. A possible explanation for the frequent violations of monotonicity might be found in Tversky and Kahneman's (1973; 1974) notion of a *representativeness heuristic*. According to this heuristic people take a cognitive short-cut such that probability assessments are biased towards the category that appears most representative. Tversky and Kahneman (1983) provide an example in which respondents report higher probabilities of seven letter words ending in _ *ing* than that of words ending in _ *n* _, whereby the latter includes the former in the sense of set-inclusion. Tversky and Kahneman argue that words ending in _ *ing* are easier to recall, and are thus presumed more common. In our question set-ups, respondents might, for example, place a higher likelihood on the temperature being between 20 and 27 degrees than on the temperature being above 20 degrees. Here, the category 20 to 27 seems the most likely outcome, and hence it is frequently given the highest probability assessment in clear violation of monotonicity with respect to inclusion.

Regardless of the specific cognitive shortcoming underlying violations of monotonicity, the apparently high number such violations makes it necessary to control for these responses when we want to determine the shape of probability judgments of a representative decision maker who can be described by the "two-stage" model.

Remark. Although the number of balls of each colour was not stated for Source III: Balls in a Bag (and in fact the distribution of colours was randomized by the manufacturer of coloured gumballs), many respondents apparently perceived the source of uncertainty as pure chance/risk given as an objective uniform distribution. Indeed, 126 (62%) respondents assumed that the balls were evenly distributed across the five colours, resulting in a very different response pattern to the other questions. The majority of the respondents gave additive responses for the balls, leaving only about a quarter of responses with inverse S-shaped probability judgments. Of these, a third were consistent with monotonicity (somewhat higher than for the true uncertainty questions). The high level of additive judgments for the balls results in inverse-S shaped probability judgments making up less than 50% of monotonicity consistent probability judgments overall, contrary to previous research where the inverse-S shaped weighting function accounts for the majority of monotonicity consistent judgments.

5.2 Distribution of inflection points

Let us now restrict attention to probability judgments that exhibit an inverse S-shape. We analyze the data for three different scenarios. The first scenario only includes probability judgments that are consistent with monotonicity. The second scenario includes only probability judgments that violate monotonicity; and the third scenario includes both those consistent with and those violating monotonicity.

We start by presenting distributions of the inflection points α predicted from the constrained linear regressions detailed in Section 4.2 above. Figure 3 shows the distribution of α for responses consistent with monotonicity; Figure 4 shows this distribution for responses violating monotonicity; and Figure 5 includes both responses consistent with and those that violate weak monotonicity.

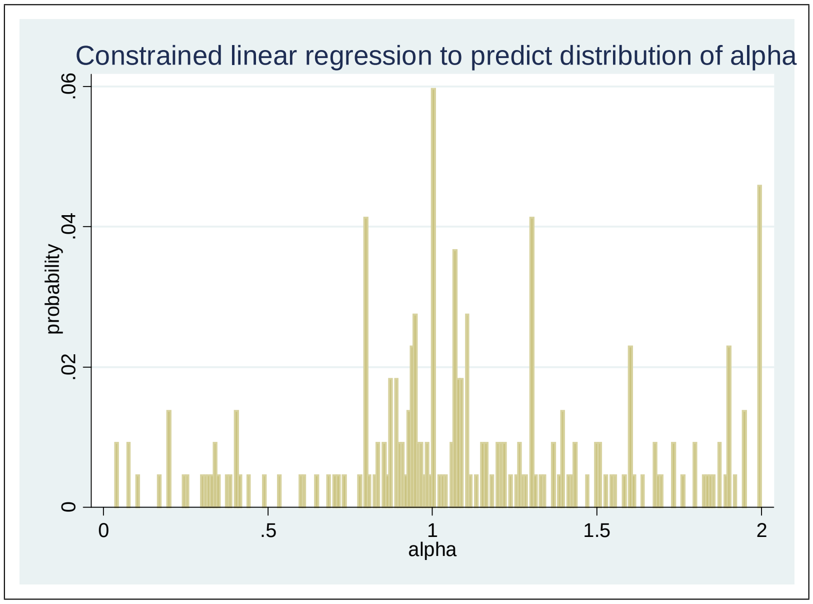

Figure 3: Distribution of alpha values: responses consistent with monotonicity

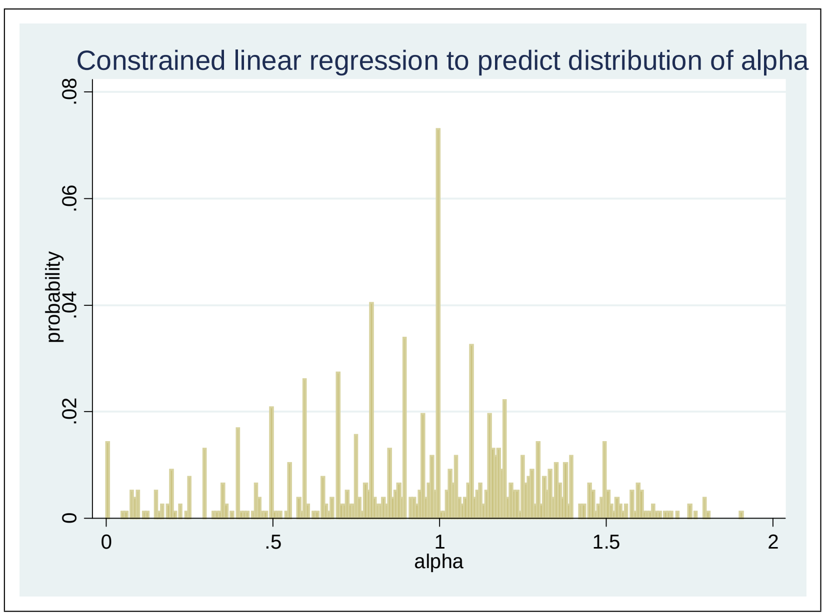

Figure 4: Distribution of alpha values: violators of monotonicity

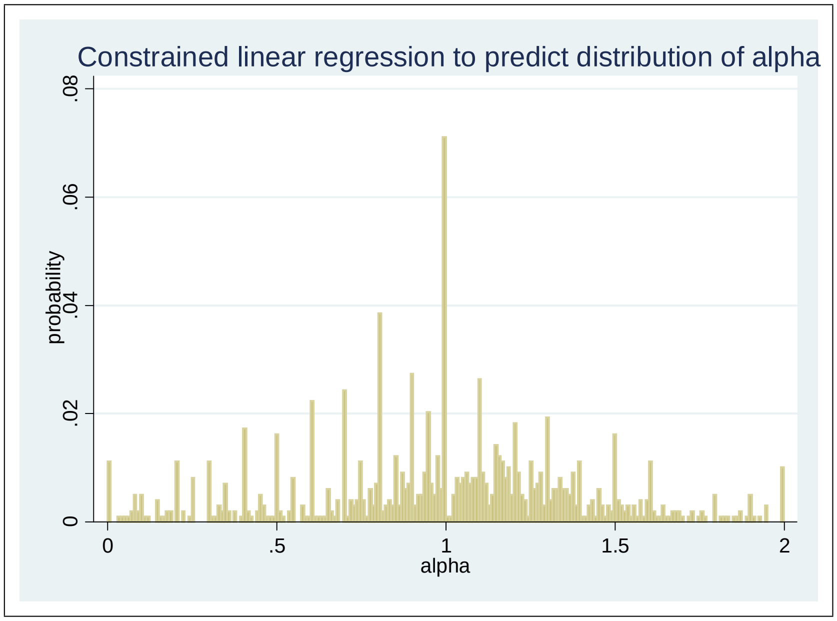

Figure 5: Distribution of alpha values: violators and non-violators of monotonicity

Table 2 summarizes means, medians and skewness for the inflection point distributions.

Table 2: Means, medians and skewness for alpha distributions

	Mean α	Median α	Skewness (p)
Monotonicity consistent (n=218)	1.12	1.07	-0.022 (0.89)
Violating monotonicity (n=767)	0.95	1.00	-0.38 (<0.001)
Consistent and violating (n=985)	0.99	1.00	-0.15 (0.05)

Figures 3-5 and Table 2 show that the inclusion of monotonicity violations in the data would bias the mean as well as the median of the inflection point distribution leftwards compared to the scenario of only monotonicity consistent responses. A Mann-Whitney U test confirms that the mean for responses consistent with monotonicity is significantly higher than the mean for responses violating monotonicity ($z = 4.35, p < 0.001$). Further, the skewness results show that whereas responses consistent with monotonicity are approximately symmetric and normally distributed (this is also true in terms of kurtosis: $p = 0.32$), responses that violate monotonicity are significantly skewed towards the left tail end of the distribution ($p < 0.001$). Given the difference in response to the Balls question versus the true uncertainty questions, we also examined the data without the

Balls question as a robustness check for the findings below: Our results were consistent with and without this data.

5.3 Empirical measures for concave and convex regions

We next compare the representative sizes of the concave and convex regions of inverse S-shaped probability judgments. Recall from Observation 4 that our measures $\mathbb{E}[\mathbb{M}_\alpha^{cave}]$, resp. $\mathbb{E}[\mathbb{M}_\alpha^{vex}]$, would coincide with the Tversky-Fox measures $\mathbb{M}^{LSA}$, resp. $\mathbb{M}^{USA}$, whenever we have the degenerate case that $\alpha = 1$ with probability 1. Although the modal value of α is for all scenarios close to 1, the probability of α taking a value in the interval $[0.95, 1.05]$ is less than 14% for all scenarios. As a consequence, α values different from one do matter to the effect that our new global measures should give a more accurate picture about region sizes than the Tversky-Fox measures.

Table 3 below reports the expected sizes $\mathbb{E}[\mathbb{M}_\alpha^{cave}]$, resp. $\mathbb{E}[\mathbb{M}_\alpha^{vex}]$, over responses consistent (inconsistent) with weak monotonicity. Mann-Whitney U test scores are presented for the comparison between the concave and convex region size in each case. Whenever the reported z-score is positive and significant, the concave region is larger than the convex region. Conversely, for a negative and significant z-score the convex region is larger than the concave region.

Table 3: Region sizes calculated using our global measures

	Concave region	Convex region	Mann-Whitney z (p)
Monotonicity consistent (n=218)	0.24	0.20	3.44 (<0.001)
Violating monotonicity (n=767)	0.52	0.60	-4.37 (<0.001)
Consistent and violating (n=985)	0.46	0.51	-1.85 (0.06)

Our findings show that the inclusion of monotonicity violating responses in the data has two substantial impacts on our global measures.[8] First, the sizes of both concave and convex regions are much smaller for the scenario in which only monotonicity consistent responses are considered. The expected values for the concave, resp. convex, region sizes are $\mathbb{E}[\mathbb{M}_\alpha^{cave}] = 0.24$, resp. $\mathbb{E}[\mathbb{M}_\alpha^{vex}] = 0.20$, for monotonicity consistent responses. In contrast, we have the sizes of $\mathbb{E}[\mathbb{M}_\alpha^{cave}] = 0.52$, resp. $\mathbb{E}[\mathbb{M}_\alpha^{vex}] = 0.60$, for the scenario which only considers monotonicity violating responses. Because of the high number of monotonicity violating responses, the third scenario, which includes both types of responses in the data, results in sizes which are still about twice as large as under the

[8] These findings are robust with respect to the inclusion or exclusion of Source III: Balls in a Bag.

monotonicity consistent scenario. This finding was to be expected for theoretical reasons (compare Figure 1).

Second, conclusions on which region (concave or convex) is dominant differ depending on whether or not violations of monotonicity are included in the data. For the scenario that only considers monotonicity consistent responses, the concave region significantly dominates the convex region whereby the Mann-Whitney U test shows that the difference in region size is significant, $p < 0.001$. Conversely, for the scenario in which only monotonicity violating responses are considered, the convex region dominates the concave region (again, the difference in region size is significant: $p < 0.001$). If both types of responses are included, the convex region still dominates the concave region due to absolutely greater sizes of these regions for the monotonicity violating scenario.

6 Discussion: Implications for economic modelling

Let us summarize the following stylized facts from our empirical study.

1. Responses typically (73%) violate additivity and are consistent with inverse S-shaped probability judgments. An exception is the ball-bag question for which many responses are consistent with additivity.

2. A large majority of inverse S-shape consistent responses violates monotonicity: 57% from all responses versus 16% of responses that are consistent with an inverse S-shape and with monotonicity.

3. If the data is restricted to monotonicity consistent responses, the concave region dominates the convex region for inverse S-shape probability judgments.

4. The picture is reversed if the data from both, monotonicity consistent and violating, responses is considered: The inclusion of monotonicity violating responses leads to a predominance of the convex over the concave region for inverse S-shape probability judgments.

5. The sizes of both the concave and convex regions are substantially greater if monotonicity violating responses are included in the data compared to the scenario where such responses are excluded.

Stylized Facts 1 and 4 are in line with the findings of previous studies (cf. Footnote 3) that did not control for monotonicity violations. Stylized Fact 2, however, strongly suggests that such a control might be necessary if we care about the shape of probability judgments that are consistent with the "two-stage" model. Finally, stylized Facts 3 and

5 confirm that there are relevant differences between the shape of probability judgments that are consistent, resp. inconsistent, with the "two-stage" CPT model.

Both authors are economists and the present study forms part of our larger research project which investigates under which circumstances existing data on probability judgments can be used as a good proxy for decision weights for which we often lack the data. For example, the US Health and Retirement Study (HRS) and the Survey of Health, Ageing and Retirement in Europe (SHARE) provide by now several waves of representative data in which respondents express probability judgments about, e.g., their survival chances. However, in contrast to choice experiments we lack for these surveys the data on the respondents' decision weights which enter, e.g., into their life-cycle consumption and savings decisions.

Economists have been keen to obtain data on probability judgments about survival chances because expected utility life-cycle models that are calibrated with objective (projected) mortality risks (i.e., *rational expectations* models) performed rather poorly with respect to empirically observed consumption and savings behavior (cf. Manski 2004 and references therein). Due to the experimental evidence against expected utility theory combined with the non-additivity of probability judgments, it looks like a good idea to use instead a CPT representative agent life-cycle model whose non-additive decision weights are given as the average over the probability judgments from the HRS or/and SHARE data. Such an approach would presume that the identity (1) reduces for the representative "two-stage" CPT agent to

$$W(E) \approx q(E) \tag{22}$$

to the effect that decision weighting under risk $w(\cdot)$ is approximately given as the identity.

As the main insight for economic modelling, our present study suggests that we should be rather cautious with the simplifying approach (22). Suppose that violations of monotonicity are negligible for the HRS and SHARE data so that we can use this data to model the probability judgments of a representative agent who is consistent with the "two-stage" CPT model. Then our empirical findings show that (22) is not most likely not adequate but that we might need, in contrast to (22), a strongly inverse S-shaped decision weighting under risk $w(\cdot)$ that blows up the sizes of the concave and the convex regions whereby it emphasizes the convex over the concave region.

7 Concluding remarks

Previous studies on the shape of probability judgments have agreed on two stylized features. First, probability judgments typically exhibit an inverse S-shape. Second, the convex region of inverse S-shaped probability judgments tends to dominate the concave region. Furthermore, probability judgments share these two features with the decision weights of a "two-stage" CPT decision maker whereby both features are slightly more pronounced for decision weights. Both features thus lend support to the notion that data on probability judgments can be used–maybe after some slight transformation emphasizing the inverse S-shape–as proxy for the CPT decision weights. As a drawback, however, previous studies did not control for monotonicity violations, which would be inconsistent with the "two-stage" CPT model.

In this paper, we have investigated the shape of probability judgments which are consistent with the "two-stage" CPT model because they obey monotonicity. While previous studies have found the majority of monotonicity consistent probability judgments to be consistent with an inverse S-shape, we find that only about 40% of monotonicity consistent judgments fit this pattern. This is driven in part by the high levels of additive probability judgments in the Balls in a bag question (Source III) where respondents largely interpreted this as a chance/risk question. Once this question is removed we see a slight majority of monotonicity consistent responses fitting the inverse-S shape (50.3%).

Further, this inverse S-shape has now become much less pronounced than for the scenario in which monotonicity violating probability judgments are included in the data. In a reversal of the second feature, we now find that the concave region of the typical monotonicity consistent inverse S-shaped probability judgment dominates its convex region. At this point we cannot offer any answers as to why monotonicity violating responses should be favoring the size of the convex over the concave region for inverse S-shaped probability judgments.

With respect to economic modelling, our analysis suggests that data on monotonicity consistent probability judgments cannot simply be substituted for the decision weights of a "two-stage" CPT representative agent. Instead, a rather strong transformation of such probability judgments might be required that blows up the sizes of the concave as well as convex regions with an emphasis on the convex region.

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Appendix A: Formal proofs

Proof of Observation 3.

Step 1. We show that concavity below $\alpha = 1 - \varepsilon$ implies ε-bounded lower subadditivity. Let

$$\begin{aligned} E &= D, \\ E' &= D' \end{aligned}$$

and observe that

$$q(E \cup E') + q(E \cap E') = q(D \cup D')$$

as well as

$$q(E) + q(E') = q(D) + q(D').$$

Consequently,

$$\begin{aligned} q(E \cup E') + q(E \cap E') &\leq q(E) + q(E') \\ &\Leftrightarrow \\ q(D \cup D') &\leq q(D) + q(D'). \end{aligned} \tag{23}$$

By concavity below α, we have that (23) applies to all disjoint $D, D' \in \Sigma$ such that

$$q(D \cup D') \leq 1 - \varepsilon.$$

□

Step 2. Now we show that convexity above $\alpha = 1 + \varepsilon'$ implies ε'-bounded upper subadditivity. Let

$$\begin{aligned} E &= S - D, \\ E' &= S - D'. \end{aligned}$$

By de Morgan's law,

$$E \cup E' = S - D \cap D'$$

so that $D \cap D' = \emptyset$ implies

$$E \cup E' = S.$$

We also have by de Morgan's law that

$$E \cap E' = S - D \cup D'.$$

Collecting expressions gives

$$\begin{aligned} q(E \cup E') + q(E \cap E') &= q(S) + q(S - D \cup D'), \\ q(E) + q(E') &= q(S - D) + q(S - D') \end{aligned}$$

implying

$$\begin{aligned} q(E \cup E') + q(E \cap E') &\geq q(E) + q(E') \\ &\Leftrightarrow \\ q(S) - q(S - D') &\geq q(S - D) - q(S - D \cup D'). \end{aligned} \tag{24}$$

Because of

$$q(S) + q(S - D \cup D') \geq \alpha,$$

(24) applies to all disjoint $D, D' \in \Sigma$ such that

$$q(S - D \cup D') \geq \varepsilon'.$$

□

Step 3. Suppose that there exists an inverse S-shaped q that satisfies ε-bounded lower SA and has inflection point $\alpha < 1 - \varepsilon$. Then there exist, by Asssumption 2, some $D, D' \in \Sigma$ such that

$$\alpha < q(D \cup D') \leq 1 - \varepsilon$$

whereby convexity above α implies

$$q(D \cup D') > q(D) + q(D');$$

a contradiction to (10). An analogous argument applies to ε'-bounded upper SA.□

Appendix B: Questionnaire

The participants were given the following instruction:

> Please give your answer to each part as a percentage likelihood.
> Consider as an example the following question:
> *What do you think is the percent chance that it will rain on the evening of 23 April 2014 at the Hatfield Campus?*
> If you're absolutely certain that it will rain on this specific date in this location, you would answer 100%, while if you are absolutely certain it will not rain, you would answer 0%. If you believe there is a 63% chance of rain, you would give this percentage as your response.

In what follows we list the questions for all seven sources as they were presented to the participants.

Source I: Inflation

What do you think is the likelihood (percent chance) that the inflation rate (CPI) in South Africa in Quarter 1 2016 will be:

a) Between 0 and 3%?
b) More than 3%?
c) Between 0 and 6%?
d) More than 6%?
e) Between 3 and 6%?

Source II: Weather

What do you think is the likelihood (percent chance) that the temperature (in degrees Celsius) at 3pm in Pretoria (as reported by the South African Weather Service: www.saws.co.za) on 29 September will be:

a) 20 or less?
b) More than 20?
c) Between 0 and 27?
d) 27 or more?
e) Between 20 and 27?

Source III: Balls in a bag

I have with me a bag containing a number of coloured balls, each of which could be red, yellow, green, orange or blue. If I draw a ball from the bag at random, what do you think is the likelihood (percent chance) that the ball drawn will be:

a) Red or yellow?
b) Neither red nor yellow?
c) Blue?
d) Not blue?
e) Green or orange?

Source IV: Rugby

In the rugby match on 27 September between South Africa and Australia, what do you think is the likelihood (percent chance) of each of the following outcomes:

f) South Africa win, draw or lose by less than 7 points?
g) Australia win by 7 points or more?
h) Australia win, draw or lose by less than 7 points?
i) South Africa win by 7 points or more?
j) The teams draw; or the winning margin is less than 7 points

Source V: The Stock Exchange

What do you think is the likelihood (percent chance) of each of the following levels for the JSE FTSE All Share index at 3pm (SA time) on 29 September (the Financial Times markets.ft.com website will be used to confirm the price level at this time):

k) Below R51,500?
l) Above R51,500?
m) Between R51,250 and R51,500?
n) Below R51,250?
o) Above R51,250?

Source VI: The Rand Exchange Rate

What do you think is the likelihood (percent chance) of each of the following levels for the Rand/US Dollar (USD) exchange rate at 3pm (SA time) on 29 September (www.xe.com will be used to determine the exchange rate at this time):

p) More than R10.25/USD?
q) More than R10.75/USD?
r) Between R10.25 and R10.75/USD?

s) Less than R10.25/USD?
t) Less than R10.75/USD?

Source VII: The Thai Baht Exchange Rate

What do you think is the likelihood (percent chance) of each of the following levels for the Thai Baht (THB)/US Dollar (USD) exchange rate at 3pm on 29 September (www.xe.com will again be the reference point for the exchange rate):

u) More than 32THB/USD?
v) More than 31THB/USD?
w) Between 31 and 32THB/USD?
x) Less than 32THB/USD?
y) Less than 31THB/USD?