Time - Varying Volatility and Risk Premia in General Equilibrium

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Abstract

This paper offers insight on the link between the nominal bond interest rates, risk premia, and economic uncertainty shocks. The analysis is carried out using a New keynesian dynamic stochastic general equilibrium (DSGE) model with recursive preferences and stochastic volatility (SV). It is shown that up to a third-order approximation, stochastic volatility has a first order effect on the level as well as the dynamics of risk premia. Moreover, stochastic volatility induces a process for the decision rules which is similar to the Autoregressive Conditional Heteroscedasticity in Mean (ARCH - M) process introduced in Engle, Lilien and Robins (1987).

The model is estimated by Simulated Method of Moments (SMM) using U.S. quarterly data. At the the SMM estimates, results show that bonds risk premia are mainly driven by the levels of technology and preferences shocks compared to the level monetary policy shock. Similarly, monetary policy shock conditional volatility has a negligible contribution to bond risk premia means and variances. On the other hand, productivity and preferences shocks conditional volatility have large effects on the term structure of interest rates and risk premia.
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1 Introduction

As documented by a large number of empirical works, the term structure of interest rates contains important economic information including agents' expectations about future interest rates and future inflation (see, for example, Frederic Minsky (1990a, 1991)). When economic agents are risk averse, the term structure of interest rates depends on private sector agents'expectations about future short-term interest rates and risk premia. Furthermore, risk premia are empirically found to be time-varying (see, Campell and Shiller, 1991) and correlated to economic uncertainty factors.

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The time-varying property of risk premia is crucial for the accuracy and usefulness of the information extracted from the term structure of interest rates since the information extracted is usually based on the assumption that risk premia are constant over time. For example, if risk premia are time - varying, a tightening monetary policy effect on long-term rates may be offset by a decline in the risk premium as it was the case in the U.S. economy between 2004 and 2006. The federal reserve gradually increased the federal funds rate by 425 basis points while long term interest rates remained surprisingly low. This behavior contrasted with movements of long term rates in past monetary policy tightenings and has been viewed by many analysts as a "conundrum". [1] In an attempt to crack this "conundrum", empirical work including Cochrane and Backus (2007), Rudebusch et al (2007) have pointed out that the risk premium may have declined in recent years to offset the increases of the federal funds rate. Similarly, Kurmann and Otrok (2011) find in a VAR framework that long-term interest rates do not respond to productivity news shocks because the responses of the risk premium part and the expectations part offset each other.

On the other hand, since risk premia are compensation for uncertainty in asset payoffs, it is crusial to understand whether different sources of uncertainty affect them in the same way if a policy maker has to respond to risk premia variations.

This paper provides a quantitative analysis of the term structure of nominal bond interest rate where risk premia are time-varying. The analysis is conducted using a New keynesian dynamic stochastic general equilibrium (DSGE) model with recursive preferences and stochastic volatility. The analysis focuses on the role played by the nature of economic shocks in the level as well as the variability of interest rates and risk premia. This is motivated by two reasons: first, empirical studies in macroeconomics and finance have pointed out that time - varying volatility is a prominent feature of the U.S. post war data and is essential to understand asset prices analysis and economic decisions under uncertainty (see, for example, Engle, 1995, Hamilton, 2010). On the other hand, Hamilton (2010) shows that misspecifying the conditional volatility in macroeconomic models can also have an impact on the mean of the variables. Second, recent studies including Rudebusch and Swanson (2010), Binsbergen, Fernandez - Villaverde, Koijen and Rubio - Ramirez (2010) have shown that DSGE models wherein households have recursive preferences can replicate business cycle and asset prices data as opposed to standard preferences.

[1] See Cochrane and Backus (2007), Rudebusch et al (2007) for this issue called the "Greenspan Conundrum" in the finance literature

The term structure of interest rates and risk premia in general equilibrium models have attracted a large amount of literature. Jerman (1998), Lettau (2003) and Uhlig (2004) among others, have studied asset prices and risk premia in real business cycle (RBC) models. Studies including Rudebusch *et al* (2008), Ravenna and Seppala (2006), Bianca De Paoli *et al* (2010), and Hordahl *et al* (2007) have analysed the implications of standard New keynesian models for the term structure of interest rates. More recently, Rudebusch and Swanson (2012), van Binsbergen, Fernandez - Villaverde, Koijen and Rubio - Ramirez (2010), Andreasen (2012), Doh (2013), analyse the term structure of interest rates in DSGE frameworks where consumers display recursive preferences. Doh (2013) estimates an endowment economy with long - run risk and stochastic volatility (SV). The findings of the paper show that time - varying term premium is more driven by inflation volatility shock than by consumption growth volatility shock contrary to previous findings. van Binsbergen, Fernandez - Villaverde, Koijen and Rubio - Ramirez (2010) extend Doh (2013) work to a production economy but with constant volatility in the shocks and exogenous inflation dynamics. The maximum likelihood estimates of their baseline model indicate large risk aversion, large capital adjustment costs, and an elasticity of intertemporal substitution (EIS) larger than one. The article by Rudebusch and Swanson (2012) use calibrated full-fledged New keynesian model with recursive preferences, firms specific capital, and long - run risk. They find that recursive preferences combined with long - run risk in monetary policy and productivity shocks are capable of replicating salient features of business cycle and asset prices simultaneously.

In this work, I focus on the contribution of volatility risk to the mean as well as the dynamics of interest rates and risk premia in a New keynesian production economy with recursive preferences and stochastic volatility. The model features real rigidities by allowing adjustment cost in capital and habit formation. Unlike in previous studies (as in Rudebusch and Swanson, 2012), the capital input in the production function is variable. More specifically, I examine the role played by each source of uncertainty - including shocks volatility uncertainty - in the determination of the level as well as the dynamics of the risk premium. This is important for economic stabilization because the results of a policy responses to exogenous shocks depend on risk premia. As a sensitivity exercise, I compare the habit formation preferences effect on the size and dynamics of bond risk premia when the capital stock is fixed and when the capital stock can be adjusted costlessly.

It is challenging to study the term structure of interest rates in a DSGE model. Especially, risk premia are difficult to compute because DSGE models are non-linear systems and analytical solutions are unavailable for the general case. Numerical methods such as value function iteration (VFI) or policy function iteration (PFI) are computationally infeasible because of the large number of state variables. Since the model does not have an exact analytical solution, I use perturbation method that involves taking a third-order expansion of the policy rules around the deterministic steady state. For detailed explanations of this approach, see Jin and Judd(2002), Schmitt-Grohé and Uribe(2004), and Kim, Kim, Schaumburg and Sims(2008), Martin Andreasen (2011). Perturbation methods deliver a zero risk-premium at first-order approximation due to the certainty equivalence property at first-order; and a constant risk-premium at second-order approximation. A third-order approximation (at least) is needed to obtain a time-varying risk premia as observed in the data.

Moreover, some parameters of the model are estimated by Simulated Method of Moments (SMM) and the remaining carefully calibrated to the U.S. economy at a quarterly frequency. SMM is an attractive method to estimate nonlinear DSGE models because, as shown by Lee and Ingram (1991), and Duffie and Singleton (1993), it delivers consistent parameter estimates like Maximum Likelihood (ML) method. In addition, it is generally robust to misspecification and the computation of the statistical objective function is quite cheap especially in models with large number of variables (see Ruge-Murcia, 2007 and 2010).

It is shown from the second - and third - order approximated solutions that, stochastic volatility induces an Autoregressive Conditionally Heteroscedastic in mean (ARCH - M) type process for the decision rules as in Engle, Lilien and Robins (1987). The difference between the ARCH - M process in this model and the purely statistical ARCH - M is that the parameters here are restricted to structural parameters and the conditional volatility is that of macroeconomic shocks instead of the conditional volatility of the decision rules themselves. It follows that the conditional volatility has a first order effect at second - order approximation and induces additional dynamics at the third - order approximation. Thus, as in Engle, Lilien and Robins (1987), the conditional volatility affects the conditional mean of the risk premium at second - and third - order approximations.

To understand the effect of the presence of stochastic volatility on the term structure of interest rates, we carry also out an estimation under which the model shocks volatilities are restricted to be homoscedatic. The SMM estimates under the benchmark (unrestricted) model show evidence of time - varying volatility in monetary policy, preferences and productivity shocks. Moreover, we find that a high risk aversion coefficient, a higher habit formation parameter and a larger capital adjustment costs are needed to match the data under the constant volatility model.

The model predicts positive risk premia leading to an upward sloping average yield curve. With regard to the levels of the shocks, the findings can be summarized as follows. The level of productivity shock has a shifting effect on the yield curve whereas monetary policy and preferences shocks affect the slope of the yield curve. For short maturities (one- to six-periods) of the yield curve, risk premia are mainly driven by technology shocks.However, the importance of technology shock decreases as the maturity increases. For longer maturity bonds, preferences shocks contribute more to risk premia than other shocks. Monetary policy shocks have only limited effect on the short - end of the yield curve. Its contribution to the size of bond risk premia declines very quickly after the three-month maturity.

As for the volatility shocks the main drivers of interest rates and risk premia are productivity and preferences volatility shocks with a limited role for monetary policy volatility shock as in the case of the level shocks. The dominant volatility shock is the productivity volatility shock. Productivity volatility shocks have a positive higher impact on the level as well as on the volatility of the long - end of the yield curve than on shorter maturities interest rates. Preferences volatility shocks, on the contrary, have a higher impact on shorter maturity interest rates. Therefore, positive productivity volatility shocks steepen the slope of the yield curve whereas positive preferences shocks flatten the yield curve. This implies that time - varying real unceratinty induces additional dynamics in the term structure of interest rates.

The effect of habit formation on risk premia depends on whether the capital stock is fixed or variable. When the capital stock is fixed, increasing the habit parameter leads increases in risk premia as found in previous work. However, when the capital stock can be adjusted costlessly, higher habit has small and decreasing effect on risk premia.

The rest of the paper is organized as follows. Section 2 presents some stylized facts on the term structure of interest rates. Section 3 describes the model and section 4 discusses the derivation of interest rates and risk premia from the equilibrium conditions as functions of macroeconomic factors. In Section 5, I present the solution method of the model. The econometric method (SMM) is explained in section 6. Finally, Section 7 discusses the implications of the model for interest rates and risk premia and presents the ressults.

2 Stylized Facts of Term Structure of Bond Interest Rates

The goal of this part is to make a quick review of some key term structure of interest rates stylized facts. I use six bond interest rates to compute selected statistics : the three-month (3m), six-month (6m), twelve-month (1y) maturity interest rates are Treasury Bill rates while the three-year (3y), five-year (5y) and ten-year (10y) maturity interest rates are Treasury constant maturity yields. The raw data used are taken from the FRED data base of the Federal Reserve Bank of St. Louis and available at their website (*www.stls.frb.org*) except the 1y interest rate series which is from Gürkaynak, Refet S., Brian Sack and Jonathan H. Wright (2007) dataset. All interest rates data are daily observations at the source from 1962 to 2007. The sample period is between 1962Q2 to 2007Q4 and is determined by the availability of the 12-month interest rate. Quarterly observations are obatained by taking the first trading day observation of the second month of each quarter (i.e February, May, August, November) instead of averaging over the quarter.[2]

Table 1: Selected Term Structure Statistics:Sample: $1962Q1 - 2001Q3$

Maturity (n)		3m	6m	1y	3y	5y	10y
Means							
	Yields (i^n)	5.48	5.63	6.02	6.42	6.61	6.83
	Excess return$(xhpr_n)$	-	18	25	102	109	178
	Spreads $(i^n - i^1)$	-	15	54	94	113	135
Standard deviations							
	Yields (i^n)	2.52	2.44	2.32	2.40	2.36	2.34
	Excess return $(xhpr_n)$	-	2.02	3.87	22.34	21.06	23.70
	Spreads $(i^n - i^1)$	-	0.22	0.43	2.01	1.87	1.7
Autocorr		0.92	0.92	0.92	0.93	0.94	0.95

Note: Yield means and all standard deviations are annualized and expressed in percent. Excess holding period returns and yield spreads means are in basis points. For each maturity n, *xhpr* is the return from holding an n-period bond one period minus the 1-period interest rate. *xhpr* are computed using the formulae: $xhpr^n_{t+1} = hpr^n_{t+1}$ -i_t where hpr^n_{t+1} = $\log(Q^{n-1}_{t+1})$- $\log(Q^n_t)$ and Q^n_t is the time t price of the n-period bond,

[2] The results of avering over the quarters differ significantly only in terms of variances. The variances of the quarterly series obtained by averaging over the quarters are significantly smaller than those of taking the first trading day of the second month of the quarter.

Table 1 summarizes some key features of the term structure. First, the term structure of interest rates is upward sloping on average over the entire sample period. The unconditional empirical means (annualized) of interest rates range from 5.48% for the three-month maturity rate to 6.83% for the ten-year rate. The average ten-year - three-month nominal interest rates spread is positive (135 basis points). This means that, on average, the slope of the yield curve is positive and that long-term rates exceed short-term rates over the sample period. Second, the volatility of the yields is decreasing with maturity meaning that the term structure of volatilities is downward sloping. However, the rate of decrease in the volatility is very low accross maturities. The three-month interest rate volatility is only 18 basis points larger than the volatility of the ten-year maturity rate. It is clear from table 1 that risk premiums, measured here by the excess holding period return ($xhpr$), are important in size as well as in variability. Excess holding period return is increasing in maturity on average and is time-varying over the sample period. Holding a ten-year bond for one period is expected to yield on average 103 basis points ($xhpr_{40} = 178\ bp$) more than the current three-month bond interest rate. Long-term risk premia are very volatile relative to short-term risk premia with the volatilities structure ranging from 2%, for the six-month risk premium, to 23.7% for the ten-year risk premium. Moreover, interest rates are very persistent with the autocorrelation coefficient of long-term maturities slightly higher than those of short-term rates.

3 The Model

The model features a standard New-keynesian economy wherein households and firms optimize. Consumers derive utility from a composite consumption good and leisure. The composite good is produced by a representative firm with a continuum of intermediate inputs goods produced by monopolistically competitive firms. Prices stickiness is modelled according to Rotemberg (1982) quadratic adjustment scheme. Consumers can save resources by using nominal bonds or capital. There is a central banker who adjusts the nominal short-term interest rate according to a Taylor-type rule.

3.1 Households

The representative household utility function features recursive preferences as in Epstein and Zin (1989). Following Rudebusch and Swanson (2012), the value function is defined as :

$$V_t = \begin{cases} u(c_t, n_t) + \beta \left(E_t V_{t+1}^{1-\varphi}\right)^{\frac{1}{1-\varphi}} & \text{if } u(c_t, n_t) > 0 \\ -u(c_t, n_t) + \beta \left(E_t (-V_{t+1})^{1-\varphi}\right)^{\frac{1}{1-\varphi}} & \text{if } u(c_t, n_t) < 0 \end{cases} \quad (1)$$

where V_t is time t value function, u is the felicity function (periodic utility function), E_t is the mathematical expectation given the time t information set, $\beta \in (0, 1)$ is the subjective discount factor, and φ is the Epstein - Zin parameter ($\varphi \in \mathbb{R}$).

The periodic utilility function features habit formation in consumption and is separable in labour. That is, u is defined as:

$$u(c_t, n_t) = d_t \left\{ \frac{(c_t - bc_{t-1})^{1-\gamma}}{1-\gamma} - \phi_0 z_t^{*(1-\gamma)} \frac{{n_t}^{1-\phi}}{1-\phi} \right\}$$

where $b \in [0, 1)$ is habit strength parameter, d_t is a preferences shock that affects both intertemporal and intratemporal conditions, ϕ_0 is a positive parameter, n_t is hours worked, ϕ captures the elasticity of labour supply parameter, c_t is a composite index of a continuum of intermediate goods, c_t^i, $i \in [0\ 1]$. As in Rudebusch and Swanson (2012), z_t^* is the trend of the economy and the the term $z_t^{*(1-\gamma)}$ assures a balance growth path and accounts for non-market labour production activities.

The composite consumption index c_t is defined by:

$$c_t = \left[\int_0^1 (c_t^i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}, \theta > 1$$

The parameter θ is the elasticity of substitution between the individual goods. As $\theta \to \infty$, intermediate goods become closer substitutes and the weaker the firm's power on these goods.

The preferences shock d_t process features stochastic volatility and is defined as:

$$\log(d_t) = \rho_d \log(d_{t-1}) + u_t^d \quad (2)$$

where $\rho_d \in (-1, 1)$ and u_t^d is the disturbances term. We allow the conditional volatility of the preferences shock to be time - varying. That is $u_t^d = \sigma_{d,t} \varepsilon_t^d$ where ε_t^d is an independently and identically distributed with mean zero and standard deviation one and $\sigma_{d,t}$ is the time t condition volatility of u_t^d. We assume that the process of $\sigma_{d,t}$ is defined by:

$$\log(\sigma_{d,t+1}) = (1 - \rho_{\sigma^d}) \log(\overline{\sigma}_d) + \rho_{\sigma^d} \log(\sigma_{d,t}) + \sigma_{\sigma^d} \zeta_{t+1}^d \quad (3)$$

where $\rho_{\sigma^d} \in (-1, 1)$, and $\overline{\sigma}_d$, σ_{σ^d} are positive parameters; ζ_t^d is an independently and identically distributed with mean zero and standard deviation one and uncorrelated with ε_t^d. Notice that modelling a process of $\log(\sigma_{d,t})$ instead of the level $\sigma_{d,t}$ itself in (3) assures that the standard deviation is always positive.

Rudebusch and Swanson (2012) find that these preferences combined with long - run risk in monetary policy and technology shocks are capable of replicating emprical asset prices along with business cycle features[3]. This is because, contrary to the constant relative risk aversion (CRRA) preferences, recursive preferences break the linkage between the risk aversion parameter and the intertemporal elasticity of substitution (IES). In the above specification (1) (without habit formation) the EIS is given by $1/\gamma$ whereas the the agent's relative risk aversion involves both parameters γ and φ. A measure of the relative risk aversion in steady state can be approximated by : $\gamma + \varphi(1-\gamma)/(1-b)$. When $\varphi = 0$, the recursive preferences specification colapses to the standard case of constant relative risk aversion (CRRA) utility function specification. When $\gamma > 1$, the lower φ, the higher the relative risk aversion and vice - versa when $\gamma < 1$.

In addition to consumption spending and labor supply, the consumer must decide how much resources to allocate in assets including investment and a range of nominal bonds of maturities indexed by $\ell = 1, \ldots, L$. Resources include labour income, capital income, and holding of the portfolio of bonds. The consumer period t budget constraint is

$$\int_0^1 \frac{p_t^i c_t^i}{P_t} di + \Upsilon_t^{-1} x_t + \sum_{\ell=1}^{L} \frac{Q_t^\ell B_t^\ell}{P_t} = \frac{W_t n_t}{P_t} + \frac{R_t k_t}{P_t} + \sum_{\ell=1}^{L} \frac{Q_t^{\ell-1} B_{t-1}^\ell}{P_t} + S_t, \tag{4}$$

where p_t^i is the price of intermediate good i, P_t is aggregate price level, x_t is investment, Q_t^ℓ and B_t^ℓ are, respectively, nominal price and holding of bond with maturity ℓ , W_t is nominal wage, R_t is nominal rental rate per unit of capital, k_t is capital and S_t lump-sum tax or transfer. Note that an ℓ-period bond at time $t-1$ becomes an $(\ell-1)$-period bond at time t. Υ_t is the relative price of investment in terms of consumption good which is assumed exogenous. The growth rate of Υ_t is deterministic and given by:

$$\log(\Upsilon_t) = \log(\mu^\Upsilon) + \log(\Upsilon_{t-1}) \tag{5}$$

[3] Other work that use recursive preferences in macro - finance models include : Hordhal, Tristani and Vestin (2008), Binsbergen, Fernandez-Villaverde, Koijen and Rubio-Ramirez (2012), Andreasen, Fernandez-Villaverde, and Rubio-Ramirez (2013)

where μ^{Υ} is the long-run gross growth rate Υ_t.

The law of motion of the capital stock is given by:

$$k_{t+1} = (1-\delta)k_t + x_t - \Gamma\left(\frac{x_t}{k_t}\right)k_t, \tag{6}$$

where $\delta \in (0,1)$ is the capital depreciation rate. Capital accumulation is subject to adjusment cost. To get one unit of capital, the agent has to invest one unit of consumption good plus an additional cost of $\Gamma\left(\frac{x_t}{k_t}\right)k_t$ which depends on the size of investment relative to current existing capital stock. $\Gamma(.)$ has the following properties: $\Gamma^{''}(.) > 0, \Gamma(\upsilon) = 0, \Gamma^{'}(\upsilon) = 0$ where υ is the steady state of the investment - capital ratio $\frac{x_t}{k_t}$. Intuitively, these properties implies that the adjustment cost depends on net investment relative to the current capital stock. For simplicity, I assume a quadratic functional form for $\Gamma(.)$ which has the above properties as in Andreasen et al (2013). That is,

$$\Gamma\left(\frac{x_t}{k_t}\right) = \frac{\kappa}{2}\left(\frac{x_t}{k_t} - \upsilon\right)^2$$

where κ is a positive parameter which controls the size of the adjustment cost. Given an investment-capital ratio $\frac{x_t}{k_t}$, the larger κ is, the higher the adjustment cost. When $\kappa = 0$, there is no adjustment and the agent can freely change the capital stock. When $\kappa = \infty$ there is an infinite adjustment cost and the agent will choose not to invest in equilibrium. Notice the existence of variable capital in the model provides the agent with an additional chanel for consumption smoothing. Higher adjustment costs in capital makes nominal bonds riskier and allows the model to generate higher bond risk premia.

In a first stage the consumer shops intermediate goods for production of the composite good. Given a level of the composite good, the consumer chooses the inputs c_t^i, $i \in [0\ 1]$ that minimize the total cost $\int_0^1 p_t^i c_t^i di$. This implies that demand for any intermediate good i is given by:

$$c_t^i = \left[\frac{p_t^i}{P_t}\right]^{-\theta} c_t,$$

where the aggregate price level P_t is given by:

$$P_t = \left[\int_0^1 (p_t^i)^{1-\theta} di\right]^{\frac{1}{1-\theta}},$$

Using the demand functions and above price expressions, it is easy to show that the quantity of composite consumption index times the aggregate price index is equal to total purchase of intermediate goods :

$$P_t c_t = \int_0^1 p_t^i c_t^i di,$$

Plugging this expression in (4), the budget constraint takes the following form:

$$c_t + \Upsilon_t^{-1} x_t + \sum_{\ell=1}^{L} \frac{Q_t^\ell B_t^\ell}{P_t} = \frac{W_t n_t}{P_t} + \frac{R_t k_t}{P_t} + \sum_{\ell=1}^{L} \frac{Q_t^{\ell-1} B_{t-1}^\ell}{P_t} + \frac{S_t}{P_t}, \tag{7}$$

Thus, household maximizes (1) subject to (7) and (6).

The first-order conditions for the consumer's problem are derived from a lagrangian problem. Following Rudebusch and Swanson (2010) the lagrangian of the consumer problem is given by:

$$V_t + E_t \sum_{t=s}^{\infty} \beta^s \left\{ \eta_{t+s} \left[V_{t+s} - u(c_{t+s}, n_{t+s}) - \beta \left(E_t ({V_{t+s+1}}^{1-\varphi}) \right)^{\frac{1}{1-\varphi}} \right] \right\} -$$

$$E_t \sum_{t=s}^{\infty} \beta^s \left\{ \lambda_{t+s} \left[c_{t+s} + \Upsilon_{t+s}^{-1} x_{t+s} + \sum_{\ell=1}^{L} \frac{Q_{t+s}^\ell B_{t+s}^\ell}{P_{t+s}} - \frac{W_{t+s} n_{t+s}}{P_{t+s}} - \frac{R_{t+s} k_{t+s}}{P_{t+s}} - \sum_{\ell=1}^{L} \frac{Q_{t+s}^{\ell-1} B_{t+s-1}^\ell}{P_{t+s}} - \frac{S_{t+s}}{P_{t+s}} \right] \right\} -$$

$$E_t \sum_{t=s}^{\infty} \beta^s \left\{ q_{t+s} \lambda_{t+s} \left[k_{t+s+1} - (1-\delta) k_{t+s} - x_{t+s} + \Gamma\left(\frac{x_{t+s}}{k_{t+s}} \right) k_{t+s} \right] \right\}$$

The first order conditions include respectively Euler equations for V_{t+1}, c_t, k_{t+1}, x_t, n_t, and $\frac{B_t^\ell}{P_t}$:

$$\beta \eta_t {V_{t+1}}^{-\varphi} \left(E_t ({V_{t+1}}^{1-\varphi}) \right)^{\frac{1}{1-\varphi}-1} - \beta E_t \eta_{t+1} = 0, \tag{8}$$

$$d_t \eta_t (c_t - b c_{t-1})^{-\gamma} - \lambda_t - \beta b E_t \left\{ d_{t+1} \eta_{t+1} (c_{t+1} - b c_t)^{-\gamma} \right\} = 0, \tag{9}$$

$$q_t = \beta E_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} \left[r_{t+1} + q_{t+1}(1-\delta) + q_{t+1} \Gamma(\frac{x_{t+1}}{k_{t+1}}) - q_{t+1} \frac{x_{t+1}}{k_{t+1}} \Gamma'(\frac{x_{t+1}}{k_{t+1}}) \right] \right\}, \tag{10}$$

$$q_t \Upsilon_t \left[1 - \Gamma' \left(\frac{x_t}{k_t} \right) \right] = 1, \tag{11}$$

$$\lambda_t w_t = \eta_t \phi_0 d_t z_t^{*(1-\gamma)} {n_t}^{-\phi}, \tag{12}$$

$$Q_t^\ell = \beta E_t \left(\frac{\lambda_{t+1}}{\lambda_t} \frac{Q_{t+1}^{\ell-1}}{\pi_{t+1}} \right) \text{,for } \ell\text{=1,2,...L}, \tag{13}$$

where λ_t and η_t are the the budget constraint (7) and the value function constraint (1) lagrangian multipliers respectively, $r_t = \frac{R_t}{P_t}$ is the real return on capital, $\pi_{t+1} = P_{t+1}/P_t$ is the gross rate of inflation between time t, and $t+1$, $w_t = \frac{W_t}{P_t}$ is real wage and q_t is the ratio of lagrangian multipliers of constraint (7) and (6), that is the Tobin's q.

3.2 Firms

Firms are of two types: a competitive final good producer and a continuum of monopolistically competitive firms indexed by $i \in [0\ 1]$ which produce intermediate goods.

3.2.1 Final Good Producer

The final good producer behaves in a perfectly competitive manner and takes as given the prices of intermediate goods and the aggregate price index when maximizing profits. Final good is produced using only individual goods y_t^i as inputs in the following production function:

$$y_t = \left[\int_0^1 (y_t^i)^{\frac{\theta-1}{\theta}} di\right]^{\frac{\theta}{\theta-1}},$$

where y_t the quantity of the final good. Profit maximization implies that demand of input i is given by:

$$y_t^i = \left[\frac{p_t^i}{P_t}\right]^{-\theta} y_t, \tag{14}$$

3.2.2 Intermediate Goods Firms and Price Setting

Each individual firm $i \in (0\ 1)$ produces a differentiated good using the same technology given by the following production function:

$$y_t^i = A_t F(K_t^i, Z_t N_t^i), \tag{15}$$

where y_t^i is ouput, K_t^i is firm i capital demand, N_t^i is labor input and the function $F(.,.)$ is constant return to scale, strictly increasing and strictly concave in both of its arguments and satisfy the Inada conditions, A_t is a neutral stationary technology shock, Z_t is a productivity trend that affects all firms in the same way. Intermediate good producing firm $i \in (0\ 1)$ hires labor and capital in perfectly competitive makets to produce its good. Firms are owned by households who receive any profit made by firms at each period. The trend productivity shock growth is deterministic and follows the process:

$$\log(Z_t) = \log(\mu^z) + \log(Z_{t-1}), \tag{16}$$

where μ^z is the unconditional growth rate of Z_t. The neutral technology shock follows the process

$$\log(A_t) = (1-\rho_a)\log(Abar) + \rho_a \log(A_{t-1}) + u_t^a, \tag{17}$$

where $\rho_a \in (-1,1)$, $\log(Abar)$ is the unconditional mean of $\log(A_t)$, and u_t^a is the disturbances term. The conditional volatility of u_t^a is time - varying. That is the disturbances term is defined as $u_t^a = \sigma_{a,t}\varepsilon_t^a$ where ε_t^a is an independently and identically distributed with mean zero and standard deviation one and $\sigma_{a,t}$ is the time t condition volatility of u_t^a. We assume that the process of $\sigma_{a,t}$ is defined by:

$$\log(\sigma_{a,t+1}) = (1-\rho_{\sigma_{\sigma^a}})\log(\overline{\sigma}_a) + \rho_{\sigma_{\sigma^a}} \log(\sigma_{a,t}) + \sigma_{\sigma^a}\zeta_{t+1}^a \tag{18}$$

where $\rho_{\sigma_{\sigma^a}}$, $\overline{\sigma}_a$, σ_{σ^a} are positive parameters; ζ_t^a is an independently and identically distributed with mean zero and standard deviation one and uncorrelated with ε_t^a.

Prices are set according to the Rotemberg (1982) model. That is, when adjusting their prices firms face a quadratic cost which is proportional to aggregate output:

$$\frac{\theta_p}{2} y_t \left(\frac{p_t^i}{p_{t-1}^i}\frac{1}{\pi_{ss}} - 1\right)^2$$

where θ_p is a positive parameter capturing the size of the prices adjustment cost and π_{ss} is steady state inflation rate. The parameter θ_p also captures the degree of nominal price rigidity. Notice that the adjustment costs increase with the size of the prices change as well as the aggregate output. In the steady state, there is no adjustment cost.

The firm $i's$ problem is to choose K_t^i, N_t^i, p_t^i to maximize discounted profits subject to its good demand function, the production technology (15) and the price setting scheme. This can be done in two steps: first choose the capital and labor input to minimize the real cost given the production function (15) and given the real wage and capital rental rates. Second choose the price to maximize the discounted real profits subject to the demand function and given the aggregate price and quantities.

The real cost minization program is:

$$\underset{K_t^i,\, N_t^i}{Min} \left[w_t N_t^i + r_t K_t^i\right]$$

s.t $y_t^i = A_t(K_t^i)^\alpha(Z_tN_t^i)^{1-\alpha}$

$$w_t = mc_t(1-\alpha)A_t(K_t^i)^\alpha {Z_t}^{1-\alpha} N_t^{i-\alpha}$$

$$r_t = mc_t\alpha A_t(K_t^i)^{\alpha-1} {Z_t}^{1-\alpha} N_t^{i1-\alpha}$$

where mc_t is the lagrangian multiplier of the production function constraint. The first order conditions imply that:

$$\frac{K_t^i}{N_t^i} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t} \tag{19}$$

Thus, all firms will choose the same capital-labor ratio. Using the above relations in the cost function, the real cost is given by:

$$Cost_t = w_tN_t^i + r_tK_t^i = mc_ty_t^i = \frac{1}{1-\alpha}w_tN_t^i$$

Use the production function and (19) to express N_t^i as a function of y_t^i, w_t, and r_t and substitute into the cost function to get:

$$Cost_t = \frac{y_t^i}{A_t}\left[\frac{w_t}{1-\alpha}\right]^{1-\alpha}\left[\frac{r_t}{\alpha}\right]^\alpha$$

The real marginal cost mc_t is equal to the derivative of the real cost with respect to y_t^i and is given as:

$$mc_t = \frac{1}{A_t}\left[\frac{w_t}{1-\alpha}\right]^{1-\alpha}\left[\frac{r_t}{\alpha}\right]^\alpha \tag{20}$$

Note that the real marginal is independent of i meaning that all firms incur the same marginal cost.

Now in the second step, firms pick their price p_t^i to maximize:

$$E_t\sum_{s=t}^{\infty}\beta^s\frac{\lambda_{t+s}}{\lambda_t}\left[\frac{p_{t+s}^i}{P_{t+s}} - mc_{t+s} - \frac{\theta_p}{2}\frac{y_{t+s}}{y_{t+s}^i}\left(\frac{p_{t+s}^i}{p_{t+s-1}^i}\frac{1}{\pi_{ss}} - 1\right)^2\right]y_{t+s}^i$$

subject to:

$$y_{t+s}^i = \left[\frac{p_{t+s}^i}{P_{t+s}}\right]^{-\theta} y_{t+s}$$

After replacing the demand function constraint in the objective function, the first order condition with respect to p_t^i is given by:

$$\frac{y_t}{P_t}\left[\frac{p_t^i}{P_t}\right]^{-\theta} - \theta\frac{y_t}{P_t}\left[\frac{p_t^i}{P_t}\right]^{-\theta-1}\left[\frac{p_t^i}{P_t} - mc_t\right] - \theta_p\frac{y_t}{p_{t-1}^i}\left[\frac{p_t^i}{p_{t-1}^i}\frac{1}{\pi_{ss}} - 1\right]\frac{1}{\pi_{ss}} +$$
$$\beta\theta_p E_t\left\{\frac{\lambda_{t+1}}{\lambda_t}\left[\frac{p_{t+1}^i}{p_t^i}\frac{1}{\pi_{ss}} - 1\right]\frac{p_{t+1}^i}{p_t^i}\frac{y_{t+1}}{p_t^i}\frac{1}{\pi_{ss}}\right\} = 0$$

Since all firms face the same demand function and marginal cost, they choose the same price in equilibrium for the same quantity of output. That is, we have a symmetric case where $p_t^i = P_t$ and $y_t^i = y_t$ $\forall t$. With the symmetry assumption, the first order condition gives the dynamics of inflation as:

$$mc_t = \frac{\theta - 1}{\theta} + \frac{\theta_p}{\theta}\frac{\pi_t}{\pi_{ss}}\left[\frac{\pi_t}{\pi_{ss}} - 1\right] - \beta\frac{\theta_p}{\theta}E_t\left\{\frac{\lambda_{t+1}}{\lambda_t}\left[\frac{\pi_t}{\pi_{ss}} - 1\right]\frac{\pi_{t+1}}{\pi_{ss}}\frac{y_{t+1}}{y_t}\right\} \tag{21}$$

The production side equilibrium conditions are given by (14) - (21).

3.3 Monetary Policy Rule and Government

The government issue the nominal bonds and is able to control the short term nominal interest rate through open market operations. Bond issues are consistent with a zero deficit. The government budget constraint is given by:

$$S_t = \sum_{\ell=1}^{L}\frac{Q_t^{\ell-1}B_{t-1}^{\ell}}{P_t} - \sum_{\ell=1}^{L}\frac{Q_t^{\ell}B_t^{\ell}}{P_t}$$

The model is closed with a Talor-type policy rule whereby the monetary authority sets the one-period nominal interest rate as a function of inflation and ouput deviations from targetted levels.

$$\frac{1+i_t}{1+i^{ss}} = \left(\frac{1+i_{t-1}}{1+i^{ss}}\right)^{\rho_i}\left(\frac{1+\pi_t}{1+\pi^{ss}}\right)^{(1-\rho_i)\gamma_\pi}\left(\frac{y_t}{z_t^*\widetilde{y}^{ss}}\right)^{(1-\rho_i)\gamma_y}\exp(mp_t) \tag{22}$$

where i_t is the time t one-period nominal bond ineterest rate, u_t^{mp} is monetary innovation, ρ_i, γ_π, γ_y are constant policy parameters, and i^{ss}, π^{ss}, $\widetilde{y}^{ss}$ are steady values of the short term nominal interest rate, inflation and the stationary level of output $\frac{y_t}{z_t^*}$ respectively. The conditional volatility of mp_t is time - varying with $mp_t = \sigma_{m,t}\varepsilon_t^{mp}$and the volatility process is defined as:

$$\log(\sigma_{m,t+1}) = (1 - \rho_{\sigma^m})\log(\overline{\sigma}_{mp}) + \rho_{\sigma^m}\log(\sigma_{m,t}) + \sigma_{\sigma^m}\zeta_{t+1}^{mp} \tag{23}$$

where $\rho_{\sigma^m} \in (-1, 1)$ and $\overline{\sigma}_{mp}$, σ_{σ^m} are positive parameters; ζ_t^{mp} is an independently and identically distributed with mean zero and standard deviation one and uncorrelated with ε_t^{mp}.

3.4 Market Clearing and Aggregation

Using the symmety assumption the aggregate output is given by:

$y_t = A_t(K_t)^\alpha(Z_tN_t)^{1-\alpha}$

In equilibrium all the markets must clear every period:

$c_t + \Upsilon_t^{-1}x_t = y_t$

$k_{t+1} = (1-\delta)k_t + x_t - \frac{\kappa}{2}\left(\frac{x_t}{k_t} - \upsilon\right)^2 k_t$

$n_t = N_t = \int_0^1 N_t^i di$

$k_t = K_t = \int_0^1 K_t^i di$

$S_t = \sum_{\ell=1}^{L} \frac{Q_t^{\ell-1}B_{t-1}^{\ell}}{P_t} - \sum_{\ell=1}^{L} \frac{Q_t^{\ell}B_t^{\ell}}{P_t}$

3.5 Stationary Equilibrium

Since there is growth in the model due to productivity and investment shock growths, we transform the system by dividing each nonstationary variable by the relevant growth rate. Following Altig, Christiano, Eichenbaum (2005) and Andreasen et al (2013) the economy technology progress trend is defined as $z_t^* = \Upsilon_t^{\frac{\alpha}{1-\alpha}} z_t$. That means that aggregate variables such as consumption, output, real wage grow at the growth rate of z_t^* whereas investment, and capital grow at the growth rate of $\Upsilon_t z_t^*$. We denote the transformed stationary variables with a ˜

The stationary system is defined as: $\widetilde{c}_t = \frac{c_t}{z_t^*}$, $\widetilde{y}_t = \frac{y_t}{z_t^*}$, $\widetilde{x}_t = \frac{x_t}{\Upsilon_t z_t^*}$, $\widetilde{w}_t = \frac{w_t}{z_t^*}$, $\widetilde{k}_{t+1} = \frac{k_t}{\Upsilon_t z_t^*}$, $\widetilde{q}_t = q_t\Upsilon_t$, $\widetilde{r}_t = r_t\Upsilon_t$, $\widetilde{V}_t = \frac{V_t}{z_t^{*1-\gamma}}$, $\widetilde{\lambda}_t = \frac{\lambda_t}{\eta_t z_t^{*-\gamma}}$. Thus, the stationary equilibrium is given by:

$$\left(\frac{E_t({V_{t+1}}^{1-\varphi})}{V_{t+1}}\right)^{\frac{\varphi}{1-\varphi}} = \frac{\eta_{t+1}}{\eta_t} \tag{24}$$

$$\widetilde{\lambda}_t = d_t(\widetilde{c}_t - b\widetilde{c}_{t-1}\frac{z_{t-1}^*}{z_t^*})^{-\gamma} - \beta b E_t\left\{d_{t+1}\frac{\eta_{t+1}}{\eta_t}(\widetilde{c}_{t+1}\frac{z_{t+1}^*}{z_t^*} - b\widetilde{c}_t)^{-\gamma}\right\}, \tag{25}$$

$$\widetilde{q}_t = \beta E_t\left\{\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\left[\frac{z_{t+1}^*}{z_t^*}\right]^{-\gamma}\frac{\eta_{t+1}}{\eta_t}\frac{\Upsilon_t}{\Upsilon_{t+1}}\left[\widetilde{r}_{t+1} + \widetilde{q}_{t+1}(1-\delta) + \widetilde{q}_{t+1}\Gamma(\frac{\widetilde{x}_{t+1}}{\widetilde{k}_{t+1}}\frac{z_{t+1}^*}{z_t^*}\frac{\Upsilon_{t+1}}{\Upsilon_t})\right]\right\} - \beta E_t\left\{\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\left[\frac{z_{t+1}^*}{z_t^*}\right]^{-\gamma}\frac{\eta_{t+1}}{\eta_t}\frac{\Upsilon_t}{\Upsilon_{t+1}}\left[\widetilde{q}_{t+1}\frac{z_{t+1}^*}{z_t^*}\frac{\Upsilon_{t+1}}{\Upsilon_t}\frac{\widetilde{x}_{t+1}}{\widetilde{k}_{t+1}}\Gamma'(\frac{\widetilde{x}_{t+1}}{\widetilde{k}_{t+1}}\frac{z_{t+1}^*}{z_t^*}\frac{\Upsilon_{t+1}}{\Upsilon_t})\right]\right\}, \tag{26}$$

$$\widetilde{q}_{t+1}\left[1-\Gamma'\left(\frac{x_t}{k_t}\frac{z_t^*}{z_{t-1}^*}\frac{\Upsilon_t}{\Upsilon_{t-1}}\right)\right]=1, \tag{27}$$

$$\widetilde{\lambda}_t\widetilde{w}_t=\phi_0 d_t {n_t}^{-\phi} \tag{28}$$

$$Q_t^\ell=\beta E_t\left(\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\left[\frac{z_{t+1}^*}{z_t^*}\right]^{-\gamma}\frac{\eta_{t+1}}{\eta_t}\frac{Q_{t+1}^{\ell-1}}{\pi_{t+1}}\right)\text{,for } \ell\text{=1,2,...L}, \tag{29}$$

$$\widetilde{w}_t=mc_t(1-\alpha)A_t(\widetilde{k}_t)^\alpha {n_t}^{-\alpha}$$

$$\widetilde{r}_t=mc_t\alpha A_t(\widetilde{k}_t)^{\alpha-1}{n_t}^{1-\alpha}$$

$$mc_t=\frac{\theta-1}{\theta}+\frac{\theta_p}{\theta}\frac{\pi_t}{\pi_{ss}}\left[\frac{\pi_t}{\pi_{ss}}-1\right]-\beta\frac{\theta_p}{\theta}E_t\left\{\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\left[\frac{z_{t+1}^*}{z_t^*}\right]^{-\gamma}\frac{\eta_{t+1}}{\eta_t}\left[\frac{\pi_t}{\pi_{ss}}-1\right]\frac{\pi_{t+1}}{\pi_{ss}}\frac{\widetilde{y}_{t+1}}{\widetilde{y}_t}\frac{z_{t+1}^*}{z_t^*}\right\} \tag{30}$$

$$\frac{1+i_t}{1+i^{ss}}=\left(\frac{1+i_{t-1}}{1+i^{ss}}\right)^{\rho_i}\left(\frac{1+\pi_t}{1+\pi^{ss}}\right)^{(1-\rho_i)\gamma_\pi}\left(\frac{\widetilde{y}_t}{\widetilde{y^{ss}}}\right)^{(1-\rho_i)\gamma_\pi}\exp(mp_t) \tag{31}$$

$$\widetilde{y}_t=\widetilde{c}_t+\widetilde{x}_t \tag{32}$$

$$k_{t+1}=(1-\delta)k_t\left[\frac{z_{t+1}^*}{z_t^*}\frac{\Upsilon_{t+1}}{\Upsilon_t}\right]^{-1}+x_t-\frac{\kappa}{2}\left(\frac{x_t}{k_t}\frac{z_t^*}{z_{t-1}^*}\frac{\Upsilon_t}{\Upsilon_{t-1}}-\upsilon\right)^2 k_t\left[\frac{z_{t+1}^*}{z_t^*}\frac{\Upsilon_{t+1}}{\Upsilon_t}\right]^{-1} \tag{33}$$

In the following section, I review the relation between the bond prices implied by the economic model and the term structure of interest rates, and define risk premia. Thus, interest rates and risk premia are derived as functions of macroeconomic fundamentals.

4 Interest Rates and Risk Premia in DSGE Models

In this section, we provide an explicit relationship between bond interest rates, risk premia and prices derived from the model. The intention is only to be explicit about the variables used in the empirical analysis. Following the literature, the interest rate (gross) of a one-period bond is given by

$$i_t^1=\frac{1}{Q_t^1} \tag{34}$$

More generally, the gross nominal interest rate of the ℓ-period bond is defined as

$$i_t^\ell = \left[Q_t^\ell\right]^{\frac{-1}{\ell}} \tag{35}$$

Here the overall risk involved in long-term nominal bonds is twofold: first, there is a risk of capital loss in the future in case of resaling the bond before the maturity date. Because the bond future prices are not known with certainty in advance, the eventual resale[4] price might be less than the purchase price. Second, there is an inflation risk involved in nominal long-term bonds because inflation can erode the bond value in the future. The risk premium can be derived recursively by rewriting the Euler equation of bonds demand as,

$$Q_t^\ell = Q_t^1 E_t\left(Q_{t+1}^{\ell-1}\right) + \beta cov_t\left(Q_{t+1}^{\ell-1}, \frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\left[\frac{z_{t+1}^*}{z_t^*}\right]^{-\gamma}\frac{\eta_{t+1}}{\eta_t}\frac{1}{\pi_{t+1}}\right), \tag{36}$$

where we used the fact that the one-period bond price is

$$Q_t^1 = E_t\left(\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\left[\frac{z_{t+1}^*}{z_t^*}\right]^{-\gamma}\frac{\eta_{t+1}}{\eta_t}\frac{1}{\pi_{t+1}}\right). \tag{37}$$

There are various formulas of risk premiums in the literature but Rudebusch *et al.* (2007) show that all definitions are highly correlated. For example, the $\ell-$period term-premium, denoted by $TP_{\ell,t}$, is usually defined as the difference between an $\ell-$period interest rate and expected average of short-term rates over the maturity period, that is,

$$TP_{\ell,t} = i_t^\ell - \frac{1}{\ell}E_t\sum_{s=0}^{\ell-1} i_{1,t+s} \tag{38}$$

In this paper, the risk premium is defined as the excess holding period return, that is, the return from holding an $\ell-$period bond for one period relative to the return of one-period bond[5]. To obtain an expression for risk premium, we rewrite (36) as

$$E_t\left[\frac{Q_{t+1}^{\ell-1}}{Q_t^\ell}\right] = \frac{1}{Q_t^1} - cov_t\left[\frac{Q_{t+1}^{\ell-1}}{Q_t^\ell}, \beta\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\left[\frac{z_{t+1}^*}{z_t^*}\right]^{-\gamma}\frac{\eta_{t+1}}{\eta_t}\frac{1}{1+\pi_{t+1}}\frac{1}{Q_t^1}\right] \tag{39}$$

[4] For example in case of a negative realization of an income shock somewhere between t and $t+\ell$, an $\ell-$period bond holder would like to redeem the bond in order to smooth its consumption

[5] Computationally, the excess holding period return requires less complementary state variables definition than the term premium

Assume an investor buying an $\ell-$period bond at time t and holds it just for one period. At time t+1, an $\ell-$period bond will be sold as an $(\ell-1)$ maturity bond. Thus, the gross return of holding an $\ell-$period for one period $H_{\ell,t+1}$ is given by:

$$H_{\ell,t+1} = \frac{Q_{t+1}^{\ell-1}}{Q_t^{\ell}}$$

Plugging the previous expression in (39) the Euler equation of the $\ell-$period bond becomes

$$E_t(H_{\ell,t+1}) = i_{1,t} + rp_t^{\ell} \tag{40}$$

where $rp_t^{\ell} = -cov_t\left[H_{\ell,t+1}, \beta\frac{\widetilde{\lambda}_{t+1}}{\widetilde{\lambda}_t}\left[\frac{z_{t+1}^*}{z_t^*}\right]^{-\gamma}\frac{\eta_{t+1}}{\eta_t}\frac{1}{1+\pi_{t+1}}i_{1,t}\right]$ is the holding period risk-premium. It is easy to show that the two definitions of risk premia are related as

$$TP_{\ell,t} = \frac{1}{\ell}E_t\sum_{s=0}^{\ell-1} rp_{t+s}^{\ell-s},$$

meaning that the term-premium is an average of all expected holding period risk-premia over the maturity period of the bond.

Equation (40) means that after ajusted for risk factor, the holding-period return is a predictor of the one-period interest rate. Note that the covariance term in the risk premium expression can either be positive or negative depending on the direction of the covariation between the holding-period return and the nominal discount factor. When high future marginal utilities- that is situations where investors need more consumption- tend to be associated with capital losses ($Q_{t+1}^{\ell-1}$ is low relative to Q_t^{ℓ} when reselling an $\ell-$period bond at t+1), investors will claim a positive risk premium for holding a long term bond instead of short-term bonds. Moreover, the two sources of risk in long-term nominal bonds highlighted above are present in the risk premium formula. First, the risk premium is affected by the comovement between the holding-period return and the real stochastic discount factor keeping the inflation rate constant. Second, correlation between the holding-period return and future inflation rate, keeping the real stochastic discount factor constant, also determines the sign and the size of the risk premia. In the first case, the resulting risk premium will be referred as the real risk premium and in the second case the inflation risk premium. The sign and the magnitude of the total risk premium will depend on the combination of these two covariance effects.

5 Model Solution

The primary focus of this paper is to understand the role played by each source of uncertainty in level and variance of interest rates and risk premia. Since the model does not have an exact analytical solution, we use a perturbation method to approximate the model given the parameters. This involves taking Taylor series expansion of the policy rules around the deterministic steady state. For detailed explanations of this approach, see Jin and Judd(2002), Schmitt-Grohé and Uribe(2004), and Kim, Kim, Schaumburg and Sims(2008). At first - order approximations, time - varying uncertainty shocks do not affect the decision rules and risk premia are equal to zero due to certainty equivalence at first-order. At second-order approximations, only the average level of shocks volatility enter in the decisions and risk premia are constant. Time-varying uncertainty effects the decisions rules and risk premium is time - varying at orders of approximation greater than three. Therefore, we solve and estimate the model at third - order approximation. The third-order approximation solution properties are provided by Martin Andreasen[6] *et* al (2013). Due to the large number of variables involving the term structure of interest rates, computing the third-order directly in Matlab requires a lot of computer memory. Thus, we use Dynare (version 4.4.2) to obtain the third-order solution.[7]

The standard approach of perturbation method writes the model general equilibrium conditions in the form:

$$E_t F(y_{t+1}, y_t, x_{t+1}, x_t) = 0 \tag{41}$$

where E_t is the conditional expectation given the time t information set, y_t is the vector of controle variables and x_t the predetermined endogenous variables and exogenous processes. F is a vectoral function of all the equilibrium conditions. In this model the controle variables vector is composed of $\widetilde{c}_t$, $\widetilde{y}_t$, $\widetilde{x}_t$, $\widetilde{w}_t$, $\widetilde{k}_{t+1}$, $\widetilde{\lambda}_t$, $\widetilde{q}_t$, $\widetilde{r}_t$, $\widetilde{V}_t$, π_t, mc_t, $\{i_t^\ell\}_{\ell=1}^{\ell=L}$, whereas the state vector contains k_t, A_t, d_t, ψ_t, mp_t, $\mu_t^z = \frac{z_t}{z_{t-1}}$, $\mu_t^\Upsilon = \frac{\Upsilon_t}{\Upsilon_{t-1}}$, σ_t^d, σ_t^a, σ_t^{mp}, c_{t-1}, i_{t-1}^1

The solution of the model is given by:

$$y_t = g(x_t, \sigma) \tag{42}$$

[6] See also Ruge - Murcia(2010)

[7] Dynare software package is available at http://www.dynare.org. For detailed explanations see Michel Julliard(2004)

$$x_{t+1} = h(x_t, \sigma) + \sigma\eta\varepsilon_{t+1} \tag{43}$$

where h and g are unknown functions, ε_t is the innovations vector of the exogenous shocks, η is constant matrix driving the variances of the innovations and σ is a scaling perturbation parameter driving the size of the uncertainty in the economy. Given that h and g are unknown, the procedure consists of approximating the functions h and g around the non-stochastic steady state point $(x, 0)$ where uncertainty is removed. Schmitt-Grohé and Uribe(2004) show that h_σ, g_σ, $h_{x\sigma}$, and $g_{x\sigma}$ evaluated at the approximated point (steady) are equal to zero. Martin Andreasen (2011) proved that at the steady state point $h_{\sigma xx} = 0$, $g_{\sigma xx} = 0$. Moreover, in the case of symmetric shocks (for example normal distribution), the terms $h_{\sigma\sigma\sigma} = 0$, $g_{\sigma\sigma\sigma} = 0$. However in the case of non-symmetric shocks (rare disaster for example), these coefficients may be different from zero.[8]

The approximate solution takes the form:

$$y_t = y + \frac{1}{2} g_{\sigma\sigma}\sigma^2 + g_x(x_t - x) + \frac{1}{2}\left[g_{xx}\right]_{\alpha_1\alpha_2}\left[x_t - x\right]^{\alpha_1}\left[x_t - x\right]^{\alpha_2} + \frac{1}{6}\left[g_{xxx}\right]_{\alpha_1\alpha_2\alpha_3}\left[x_t - x\right]^{\alpha_1}\left[x_t - x\right]^{\alpha_2}\left[x_t - x\right]^{\alpha_3} + \frac{3}{6}\left[g_{\sigma\sigma x}\right]_{\alpha_3}\sigma^2\left[x_t - x\right]^{\alpha_3} + \frac{1}{6} g_{\sigma\sigma\sigma}\sigma^3 \tag{44}$$

$$x_{t+1} = x + \frac{1}{2} h_{\sigma\sigma}\sigma^2 + \frac{1}{6} h_{\sigma\sigma\sigma}\sigma^3 + h_x(x_t - x) + \frac{1}{2}\left[h_{xx}\right]_{\alpha_1\alpha_2}\left[x_t - x\right]^{\alpha_1}\left[x_t - x\right]^{\alpha_2} + \frac{1}{6}\left[h_{xxx}\right]_{\alpha_1\alpha_2\alpha_3}\left[x_t - x\right]^{\alpha_1}\left[x_t - x\right]^{\alpha_2}\left[x_t - x\right]^{\alpha_3} + \frac{3}{6}\left[h_{\sigma\sigma x}\right]_{\alpha_3}\sigma^2\left[x_t - x\right]^{\alpha_3} + \sigma\eta\varepsilon_{t+1} \tag{45}$$

where $x = h(x, 0)$ and $y = g(x, 0) = g(h(x, 0), 0)$ and n_y and n_x are the number of controle and state variables respectively, $\alpha_1, \alpha_2, \alpha_3 = 1, ...n_x$. g_x, h_x, g_{xx}, h_{xx}, g_{xxx}, h_{xxx} $h_{\sigma\sigma}$, , $g_{\sigma\sigma}$, $h_{\sigma\sigma x}$, $g_{\sigma\sigma x}$, $h_{\sigma\sigma\sigma}$, $g_{\sigma\sigma\sigma}$ are constant coefficients standing for first, second, and third derivatives of g and h with respect to x and σ evaluated at the deterministic steady state. Notice that these coefficients are functions of the structural parameters of the model and that the parameter σ enters the decision rules as an argument capturing the risk factors. Also, the conditional volatilities of the innovations in the state vector are time - varying and enter directly in the decision rules.

[8] These results are also shown in Ruge - Murcia (2012)

Since the third order solution is computed using Dynare, the decision rules are expressed as functions of $(x_{t-1}, \varepsilon_t, \sigma)$ instead of (x_t, σ). Notice that the above representation (42) and (43) of the solution can be recovered from the Dynare representation by redefining the state vector as $v_t = (x_{t-1}, \varepsilon_t)$ as in Andreasen et al (2013). Then, it is easy to show that the equilibrium solution is expressed as:

$$y_t = g(v_t, \sigma)$$

$$v_{t+1} = \overline{h}(v_t, \sigma) + \sigma \tilde{\eta} \varepsilon_{t+1}$$

where $\overline{h}(v_t, \sigma) = (h(v_t, \sigma), 0)'$ and $\tilde{\eta} = (\eta, 0)'$.

6 Econometric Analysis

6.1 Data

The model is estimated using U.S. macroeconomic as well as term structure data at the quarterly frequency. The sample period is 1962 Q1 -2007 Q4 and is limited by the availability of the term structure data.

The macro data used are *per capita* real consumption growth, *per capita* real investment growth, real wage inflation rate, *per capita* hours worked, and Consumer Price Index (CPI) inflation rate. Consumption is NIPA measures of personal consumption expenditure on non durable goods and services. Investment is measured by private fixed nonresidential investment and expenditure on nondurable goods. *Per capita* real investment and consumption are obtained by dividing these variables by the quarterly CPI and the Bureau of Economic Analysis (BEA) estimate of the mid-month U.S. population. Hours worked is the average weekly hours of production and nonsupervisory employees in the manufacturing sector. Since the time endownment is normalized to one in the model, we assume a time endownment of 120 ($5\times$ 24) which corresponds to five working days per week and divide each obsevation of the original hours worked series by 120. All the macro data are taken from the Federal Reserve Bank of St. Louis website (*www.stls.frb.org*) and are seasonally adjusted at the source.

The term structure of interest rates data are the three-month nominal interest rate, the ten-year nominal interest rate as well as the ten - year excess holding period return. Since a period corresponds to a quarter in the model, the empirical counterparts of the one-period and forty-period interest rates are respectively three-month and ten-year interest rates. Thus, the model counterparts of the three term structure series are i_t^1, i_t^{40} and rp_t^{40}. The three-month rate is treasury bill whereas the ten-year interest rate is constant maturity rate. Both interest rate series are taken from the Federal Reserve Bank of St. Louis website. The original interest rates were available at a daily frequency. Quarterly observations have been obtained by taking the first trading day observation of the second month of each quarter[9] (February, May, August, November). These interest rates are well suited to the theoretical interest rates of the model because first, they are interest rates on zero-coupon bonds and second the default risk is negligible. Excess holding period return series is computed using continuously - compounded yields from Gürkaynak, Refet S., Brian Sack and Jonathan H. Wright (2007) dataset. The ten - year excess holding period return is used as a proxy for risk premium whereas the inclusing of both three - month and ten - year interest rates captures the slope of the yield curve.

In all, eight data series have been used in the estimation.

6.2 Paramaters Estimation: Simulated Method of Moments (SMM)

The parameters of the model are estimated by Simulated Method of Moments (SMM). SMM consists in minimizing a weighting distance between unconditional moments predicted by the model and the corresponding data moments counterparts. Basically, the predicted moments are based on artificial data simulated from the model while data moments are directly computed from actual data. Consider the full DSGE model with unknown $k \times 1$ parameters vector denoted by $\theta \in \Theta$. Suppose, we have T observations of stationary and ergodic economic data series $\{q_t\}$. Let's denote by

$\frac{1}{T}\sum_{t=1}^{T} m(q_t)$ a set of p moments computed from the data where $p \geq k$[10]. For given values of parameters θ we can compute the same set of moments from artificial data simulated from the model. Assume that the sample size of the synthetic series is $\tau \times T$ and denote these moments by:

$\frac{1}{T\tau}\sum_{t=1}^{T\tau} m(q_t(\theta))$ where $\tau \geq 1$ is an integer.

The SMM estimator of θ is defined by:

$$\widehat{\theta}_{SMM} = \underset{\theta\in\Theta}{\arg\max} M(\theta)' \; W \; M(\theta)$$

[9] Instead of averaging over the quarter

[10] A necessary condition for identification

where $M(\theta) = \frac{1}{T}\sum_{t=1}^{T} m(q_t) - \frac{1}{T\tau}\sum_{t=1}^{T\tau} m(q_t(\theta))$ and W is a $p \times p$ positive-definite weighting matrix. Thus, the SMM estimator $\widehat{\theta}_{SMM}$ is the value of the parameters vector θ that minimizes the distance between a set of data moments and those implied by the model. As Shown in Ruge - Murcia, the asymptomic distribution of $\widehat{\theta}_{SMM}$ is normal and the asymptotic variance matrix is given by:

$$(1 + \frac{1}{\tau}) \left(J^{'}WJ\right)^{-1} J^{'}WSWJ \left(J^{'}WJ\right)^{-1} \tag{46}$$

where $J = E\left(\frac{\partial m(q_t(\theta))}{\partial \theta}\right)$ and S is the long-run variance matrix of the sample moments vector. Notice that when the number of simulated samples $\tau \to +\infty$, the SMM asymptotic variance matrix converges to that of the Generalized Method of Moments (GMM).

SMM is an attractive method to estimate nonlinear DSGE models because, it delivers consistent parameter estimates (see, Lee and Ingram (1991), Duffie and Singleton (1993)). Moreover, as shown in Ruge-Murcia (2007), SMM is generally robust to misspecification and the computation of the statistical objective function is quite cheap. Ruge-Murcia (2010) explains in detail the application of SMM for the estimation of higher-order DSGE models and provides Monte-Carlo evidence on its small-sample properties. The standard errors of the estimates have been computed using the asymptotic variance matrix in (46). However, this is just an approximation as Ruge-Murcia (2010) shows that SMM based asymptotic standard errors tend to overstate the actual variability of the estimates. A non - parametric boostrap type standard errors are computationally expensive at third - order approximation with a large number of variables like in this model.

Artificial data are obtained by simulating the model based on the prunned version of the third-order approximate solution proposed by Martin Andreasen, Jesús Fernández-Villaverde and Juan Rubio-Ramírez (2013). The innovations are drawn from the normal distribution for the simulation. The number of moments used in the estimation is thirty two: the variances, first- and second-order autocovariances as well as the unconditional means of the eight data series. Thus, thirty-two moments are used in the estimation of the parameters. The weighting matrix used is the diagonal of Newey-West estimator of long-run variances of the moments with a Bartlett kernel and bandwidth given by the integer of $4(\mathrm{T}/100)^{2/9}$ where T is the sample size. The sample size here is T=182 which implied a bandwith value of 4.569. The number of the simulated observations is ten times the sample size T as suggested in Ruge-Murcia (2010).

The number of estimated parameters is twenty : five preferences parameters $\beta, b, \gamma, \varphi, \phi$; five shock levels parameters including the persistence (ρ_a, ρ_d) and unconditional standard deviation $(\overline{\sigma}_a, \overline{\sigma}_d, \overline{\sigma}_{mp})$ parameters of productivity, preferences and monetary policy shocks respectively; six conditional volatility shocks parameters including the persistence $(\rho_{\sigma_a}, \rho_{\sigma_d}, \rho_{\sigma_m})$ and standard deviation $(\sigma_{\sigma_a}, \sigma_{\sigma_d}, \sigma_{\sigma_m})$ parameters of productivity, preferences and monetary policy shocks, respectively; three monetary policy reaction parameters, $\gamma_\pi, \gamma_y, \rho_i$; and the capital adjustment cost parameter κ. Thus $\theta = \left[\beta, b, \gamma, \varphi, \phi, \kappa, \gamma_\pi, \gamma_y, \rho_i, \rho_a, \rho_d, \rho_m, \rho_{\sigma_a}, \rho_{\sigma_d}, \rho_{\sigma_m}, \overline{\sigma}_a, \overline{\sigma}_d, \overline{\sigma}_{mp}, \sigma_{\sigma_a}, \sigma_{\sigma_d}, \sigma_{\sigma_m}\right]'$. Since the number of moments used is thirty - two, the number of degree of freedom is twelve $(= 32 - 20)$. The remaining parameters are difficult to identify and thus have been calibrated in the next subsection.

Because the theoretical properties of SMM estimates are valid under stationnarity assumptions, a unit root test has been performed on the series used in the estimation. To this end, I use an Augmented-Dickey Fuller (ADF) and a Phillips-Perron (PP) unit root tests. The null hypothesis of unit root can be rejected at 5% level under both tests for all series except the inflation rate. However, for the inflation rate, the unit root hypothesis can be rejected at the 5% level under the PP test but cannot be rejected under the ADF test. But the ADF-statistic is -2.38 whereas the critical value is -2.39. So, I suppose that the inflation rate is also stationnary.

6.3 Calibration

During the estimation, the remaining model parameters have been calibrated as follows:

The production function parameter is set at $\alpha = 0.3$ to match the share of capital income in the U.S. data. Notice that in the model, the unconditional growth rate of consumption is given by the unconditional growth rate of the economy technology progress z_t^*, which from the definition of z_t^*, is given by: $\log(\mu^{z^*}) = \log(\mu^z) + \frac{\alpha}{1-\alpha}\log(\mu^\Upsilon)$ where μ^z and μ^Υ are the unconditional growth rate of z and Υ, respectively. The unconditional growth rate of investment is given by: $\log(\mu^{z^*})+ \log(\mu^\Upsilon)$. Thus, given α, μ^z and μ^Υ are calibrated to match the sample growth rates of consumption (1.005045) and investment (1.006068).

The disutility parameter ϕ_0 is calibrated to match a steady state hours worked of $n_{ss} = 0.34$ as in the data.

From the capital accumulation equation, the depreciation rate of capital is set as $\delta = 1-(1-\frac{x_{ss}}{kss})\mu^{z^*}\mu^{\Upsilon}$. The investment - capital ratio $\frac{x_{ss}}{kss}$ is fixed at 0.025, and given μ^{z^*}, μ^{Υ}, $\delta = 0.02$. The parameter υ in the capital adjustment cost function is then set such that there is no adjustment cost in the steady state. That is, $\upsilon = \frac{x_{ss}}{kss}\mu^{z^*}\mu^{\Upsilon}$.

Since there is no prices adjustment cost in the steady state, the model steady state mark-up Ψ is given by the expression $\Psi = \frac{\theta}{\theta-1}$. θ is set such that the long - run mark - up (gross) $\Psi = 1.1$; that is, $\theta = 11$.

The Rotemberg (1982) prices adjustment cost parameter θ_p is set such that the first order inflation dynamics is equivalent to that of a model with Calvo (1983) pricing. That is, $\theta_p = \frac{(\theta-1)\eta}{(1-\theta)(1-\beta\theta)}$ where η is the Calvo parameter, θ is the elasticity of substitution among goods and β is the subjective discount factor. The Calvo parameter is set at 0.75 to match an average price duration of 4 quarters and the subjective discount factor β is estmated.

The steady state of gross inflation rate π_{ss} is fixed as 1.008 to account for an annualized long - run inflation rate of 3.2%. The calibrated parameters are reported in table 3.

6.4 Parameters Estimates

Table 4 reports the SMM estimates of the parameters. For the sake of comaparison we also report the estimates of the parameters under the restricted model of hmoscedastic shocks. The first column reports the estimates when the conditional variances of the shocks follow stochastic voaltility processes and the second column when shock volatilities are constant.

Results under stochastic volatility (in column 1) show that there is evidence of time - varying volatility in productivity, preferences and monetary policy shocks. The estimates indicate that productivity and preferences shocks are very persistent and volatile. The autocorrelation coefficient of productivity shock is $\rho_a = 0.948$ and the unconditional standard deviation is $\overline{\sigma}_a = 0.012$. The conditonal volatility of the productivity shock is also very persistent - with an autocorrelation coefficient of $\rho_{\sigma_a} = 0.8$ - and volatile ($\sigma_{\sigma_a} = 0.42$). These estimates are similar to the estimates reported in Justiniano and Primiceri (2008).

The preferences shock is highly persistent ($\rho_d = 0.982$) and volatile ($\overline{\sigma}_d = 0.014$). The conditional volatility is moderately persistent ($\rho_{\sigma_d} = 0.605$) and less volatile than the productivity shock ($\sigma_{\sigma_d} = 0.4$).

The monetary policy shock has been contrained to an *i.i.d.* process and the estimated unconditional standard deviation is large and statistically significant ($\overline{\sigma}_{mp} = 0.001$). The conditional volatility autocorrelation coefficient is small ($\rho_{\sigma_m} = 0.4$) and not statistically different from zero but the unconditional standard deviation of the innovations ($\sigma_{\sigma_m} = 0.001$) is significantly different from zero.

The preferences parameters are in line with those reported in the literature. The subjective discount factor is $\beta = 0.9926$. There is evidence of moderate habit formation ($b = 0.57$) which is slightly lower than the standard reported value of 0.65. The estimates of the consumption curvature parameter is $\gamma = 1.57$. The Epstein - Zin parameter φ which is crucial for the relative risk aversion is estimated to be -167 which is higher than the reported value of -194 in Andreasen et al (2013). The interpretation of a negative Epstein - Zin parameter is that agents prefer to rather solve today an expected future uncertainty. This implies that that any change in expected future volatility will affect today agents decision. Notice that the elasticity of intertemporal substitution ($1/\gamma = 0.64$) is less than one with a very high risk aversion. The estimates of the Frisch elasticity of labour supply is less than one ($1/\phi = 0.15$).

The capital adjustment cost parameter estimates is moderate ($\kappa = 3.57$). The central bank reaction to deviations of inflation from the long - run inflation is higher ($\gamma_\pi = 3.225$) than its reaction to deviations of output from the steady state ($\gamma_y = 0.430$). The policy rate displays inertia with a moderate short - term interest rate smoothing parameter of $\gamma_R = 0.67$.

Now we turn to compare the results of the estimations under the benchmark model and under the restricted assumption of constant volatility. In general, the main differences between the two models reside in the estimates of the risk aversion parameters, moneteray policy shocks, and the real rigidities. The Epstein - Zin parameter under the constant volatility of shocks implies a higher relative risk aversion ($\varphi = -199.35$). The consumption curvature parameter is also slightly higher ($\gamma = 1.79$) as well as the habit formation parameter ($b = 0.77$). The most sticking difference is on the monetary policy shock. Under the constant volatility case, the standard deviation of the monetary policy shock is very small ($\overline{\sigma}_{mp} = 1.47 \times 10^{-5}$) and not statistically different from zero. It means in this case that the dynamics of the model is only driven by productivity and preferences shocks. The difference between the two set of estimates under the time - varying and constant volatility outlines the claim by Hamilton (2010) that a conditional variance misspecification has a first order effect on the conditional means.

7 Results

We present below the implications of the second - and third-order approximate solution of the model for interest rates and risk premia and perform some sensitivity exercises.

7.1 Implications for the Term Structure

To understand the implications of the model for the term structure of interest rates and risk premia we use the third - order approximated solution in (44) to express the interest rates decision rules in the following form:

$$\widehat{i}_t^{\ell} = i_v \left[\widehat{v}_t\right] + \frac{1}{2}\left[i_{vv}^{\ell}\right]_{\alpha_1\alpha_2}\left[\widehat{v}_t\right]^{\alpha_1}\left[\widehat{v}_t\right]^{\alpha_2} + \frac{1}{6}\left[i_{vvv}^{\ell}\right]_{\alpha_1\alpha_2\alpha_3}\left[\widehat{v}_t\right]^{\alpha_1}\left[\widehat{v}_t\right]^{\alpha_2}\left[\widehat{v}_t\right]^{\alpha_3} + \frac{3}{6}\left[i_{\sigma\sigma v}^{\ell}\right]_{\alpha_3}\sigma^2\left[\widehat{v}_t\right]^{\alpha_3} + \frac{1}{2}i_{\sigma\sigma}^{\ell}\sigma^2 + \frac{1}{6}i_{\sigma\sigma\sigma}^{\ell}\sigma^3 \quad (47)$$

where $\widehat{i}_t^{\ell}$ is the log deviation from steady state of the ℓ-period maturity bond interest rate; $\widehat{v}_t = (\widehat{x}_{t-1}, \varepsilon_t)$ and $\widehat{x}_{t-1}$ is a vector of log deviation of the state variables from the steady state. When shocks are symmetric, the last term $\frac{1}{6}i_{\sigma\sigma\sigma}^{\ell}\sigma^3 = 0$ which implies that third order approximation has no impact on the mean of interest rates since the constant term is equal to the constant term of the second order approximation. Notice that $\widehat{v}_t$ contains the time $t-1$ volatilities of the shocks (σ_{t-1}^d, σ_{t-1}^a, σ_{t-1}^{mp}) through the vector $\widehat{x}_{t-1}$ as well as their time t innovations. It means that the time - varying volatilities enter the decision rules as state variables and provide an additional dynamics to the term structure of interest rates. We now present below the model implications of stochastic volatility for the term structure of interest rates.

First, we compare the prediction of the model with the data by plotting selected term structure of interest rates moments computed from the model against their data counterparts in Figure 1. Panel A plots the model predicted moments againts the data counterparts whereas panel B and C display the unconditional means and standard deviations implied by the model respectively. The moments used in panel A are the unconditional means and standard deviations of interest rates at different maturities. The model moments are computed based on 150000 simulated observations.[11] All moments are transformed in percentage and annualized. The horizontal

[11] We simulate 200000 observations from which we discard the first 50000 observations.

axis is the simulated moments whereas the vertical axis is the data moments counterparts. The selected maturities are the 3m, 6m, 1y, 2y, 3y, 5y, and 10y. Recall that only the three - month and the ten - year interest rates (the red dots in figure 1) were targeted in the estimation. As panel A of figure 1 shows the model was able to match relatively well the means of the three - month and ten - year interest rates. However, the model implied standard deviation of the three - month rate is way higher than its data counterparts (4.45 vs 2.51). In panel B and C the horizontal line plots the maturity of the bonds whereas the vertical line is the values of the variables in annualized percentage. As is clear in panel B, the model was able to generate an upward sloping unconditional yield curve with long - term interest rates higher than short - term rates on average as in the data. Moreover, the standard deviations (see panel C) are decreasing across maturities which is also in line with the data. Figure 2 plot 4000 observations simulated from the model. It is evident from figure 3 that there is a lot of variations in the model generated risk premia and that long - term interest rates are smoother than short - term rates. In all, the model qualitatively is in line with the data with regard to the first and second moments of the term structure of interest rates.

Figure 3, 4 and 5 plot the responses of the term structure to a positive one standard deviation of the levels of the shocks and figure 6, 7 and 8 present the responses of the term structure to their corresponding volatility shocks. The horizontal line is the time after a shock hits the economy whereas the vertical axis measures the response of each variable. Responses to the level of the shocks refer here to responses of the system to innovations of the level of the shocks keeping the conditional volatility fixed and responses to volatility shocks refer to responses of the system keeping the levels of the shocks unchanged. That is, we examine the responses of the system to first and second moments innovations of the shocks.

With regard to the levels shocks, a positive productivity shock entails a decrease in interest rates at all maturities (figure 3). This negative effect is slightly more pronounced for shorter term rates than for longer maturities at the impact time. It means that at the impact time the spread between long - term and short - term rates positive but negligible. Thus this will tend to shift downward the yield curve. The effect of a positive productivity level shock on risk premia differs across maturities. Shorter maturity risk premia slightly increase whereas long - term premia tend to decrease. However, the order of the magnitude of the impact is small (10^{-5}). On the other hand, a positive preferences level shock has a negative impact on interest rates (figure 4). Short - term interest rates decrease more than long - term rates. Remember, the preferences shock affects directly the consumer intertemporal decisions and the model pricing kernel. Here at the SMM parameter estimates, a positive preferences shock leads to increases in the pricing kernel and bond prices which means a decrease in interest rates. The impact of a positive preferences shock on risk premia is positive and is increasing with the maturity. Long - term risk premia increase more than shorter term premia. Notice that compare to the technology level shock the magnitude of the impact of the preferences level shock is higher. As expected a positive monetary policy level shock increases interest rates of all maturities and the impact decreases with maturities as short - term rates increase more than long - term rates (figure 5). The effect vanishes quickly because the persistence parameter of the policy shock were set to zero in the model. The 6m risk premium responds negatively to a positive monetary policy shock whereas other maturity premia increase. However, the magnitude of the impact is very small compare to technology and preferences level shocks.

Now, we examine the effects of the three volatility shocks on the term structure. As figure 6 shows, a positive shock to the conditional volatility of technology leads to a decrease short - term rates and an increase in long - term rates. Risk premia respond positively by increasing and the impact is increasing with the maturity. For example, a one standard deviation in the conditional volatility of productivity leads to a more than 20 basis points (annualized) increase in the 10y bond risk premium. Notice that this impact is more important in magnifitude than the level shock effects we explored above. As it is also the case for the technology volatility shock, an increase in the conditional volatility of preferences shock leads to a precautionary behavior of the consumer as consumption decreases and investment increases[12]. However, an increase in the preferences shock volatility leads rather to a decrease in interest rates and risk premia. Long - term interest rates decrease less than short - term rates. Remember the risk premium here is the excess holding period return, that is, the expected return of holding a bond for one - period minus the current one - period bond yield. Since interest rates decrease it means that current bond prices increase and future bond prices are expected to increase more than the current price increases. Relative to the current decrease in the one - period interest rate, the expected return

[12] See the impulse responses of the macro variables in appendix. Here we focus only on analyzing the term structure

of holding a bond for a period is higher and thus agents demand less risk premium to hold longer maturity bonds. Monetary policy volatility shock has a negligible effect on the term structure of interest rates and risk premia as the impact is positive but very small (see figure 8).

7.2 Risk Premia and Volatility Shocks

In this section, we analyse the implications of the model solution to further understand how the volatility shocks affect the levels as well as the variance of risk premia. Since risk premia are compensations for uncertainty, only state variables which involve exposure to uncertainty enter in their formulae. That is, the risk premium decision rules contain terms that involve cross products of volatilities or innovations and potentially the other state variables. For example, the second-order approximation solution will deliver a constant risk premium involving constant structural parameters scaled by the volatilities of the shocks. Thus, the second-order approximation risk-premium denoted by $rp^{\ell 2rd}$ can be written[13] as

$$rp^{\ell 2rd} = \frac{1}{2}rp^{\ell,a}\overline{\sigma}_a^2 + \frac{1}{2}rp^{\ell,d}\overline{\sigma}_d^2 + \frac{1}{2}rp^{\ell,mp}\overline{\sigma}_{mp}^2 + \\ \frac{1}{2}rp^{\ell,\sigma_a}(\sigma_{\sigma^a})^2 + \frac{1}{2}rp^{\ell,\sigma_d}(\sigma_{\sigma^d})^2 + \frac{1}{2}rp^{\ell,\sigma_m}(\sigma_{\sigma^m})^2 \quad (48)$$

where $rp^{\ell,d}$, $rp^{\ell,a}$, $rp^{\ell,mp}$, rp^{ℓ,σ_d}, rp^{ℓ,σ_a}, rp^{ℓ,σ_m} are functions of structural parameters and $\overline{\sigma}_d^2$, $\overline{\sigma}_a^2$, $\overline{\sigma}_{mp}^2$ are the unconditional volatility of preferences, productivity, and monetary policy shocks respectively and σ_{σ^d}, σ_{σ^a}, σ_{σ^m} are the standard deviations of their respective innovations. Notice that the first line of (48) is the risk premium when shocks display constant volatility and the second line take into account the uncertainty involved in the conditional volatility of the shocks. Thus, compare to the constant volatility of shocks case, time - varying volatility has a first order effect and affects the conditional mean of risk premia. Imsyn this model, the constant volatility case is obtained by imposing $\sigma_j = 0$ where $j = \sigma^d$, σ^a σ^m.

At a third - order approximation, risk premia are time - varying as long as the coefficients $g_{\sigma\sigma v}$ corresponding to the risk premium decision rules in (44) are different from zero. When the volatilities of the shocks display time variation, this adds more dynamics to the risk premia since the state vector now includes conditional volatilities. In the case of non-symmetric shocks ($g_{\sigma\sigma\sigma} \neq 0$), third-order approxima-

[13] With the perturbation parameter σ fixed at 1

tions may affect the level of risk premia. Thus, the model risk premium implied by a third-order approximation denoted by $rp_t^{\ell 3rd}$ takes the form:

$$rp_t^{\ell 3rd} = rp^{\ell 2rd} + \frac{1}{6} rp_{\sigma\sigma\sigma}^{\ell,a} \overline{\sigma}_a^3 + \frac{1}{6} rp_{\sigma\sigma\sigma}^{\ell,d} \overline{\sigma}_d^3 + \frac{1}{6} rp_{\sigma\sigma\sigma}^{\ell,mp} \overline{\sigma}_{mp}^3 +$$
$$\frac{3}{6} \left[rp_{\sigma\sigma,v^-}^{\ell} \right]_{\alpha_3} \left[\widehat{v^-}_t \right]^{\alpha_3} +$$
$$\frac{3}{6} \left\{ \left[rp_{\sigma\sigma,\sigma_a}^{\ell} \right] \widehat{\sigma}_{a,t-1} + \left[rp_{\sigma\sigma,\sigma_d}^{\ell} \right] \widehat{\sigma}_{d,t-1} + \left[rp_{\sigma\sigma,\sigma_m}^{\ell} \right] \widehat{\sigma}_{m,t-1} \right\} +$$
$$\frac{3}{6} \left\{ \left[rp_{\sigma\sigma,\zeta^a}^{\ell} \right] \zeta_t^a + \left[rp_{\sigma\sigma,\zeta^d}^{\ell} \right] \zeta_t^d + \left[rp_{\sigma\sigma,\zeta^{mp}}^{\ell} \right] \zeta_t^{mp} \right\} \quad (49)$$

where $rp_t^{\ell 3rd}$ is the time t risk-premium on the ℓ-perriod bond, $rp_{\sigma\sigma,\sigma_a}^{\ell}$, $rp_{\sigma\sigma,\sigma_d}^{\ell}$, $rp_{\sigma\sigma,\sigma_m}^{\ell}$, $rp_{\sigma\sigma,\zeta^a}^{\ell}$, $rp_{\sigma\sigma,\zeta^d}^{\ell}$, $rp_{\sigma\sigma,\zeta^m}^{\ell}$ are the third - order partial derivatives of $rp_t^{\ell 3rd}$ with respect to σ^2 and σ_a, σ_d, σ_m, ζ^a, ζ^d, ζ^m respectively. v_t^- is the vector of the remaining state variables in v_t excluding $\sigma_{a,t-1}$, $\sigma_{d,t-1}$, $\sigma_{m,t-1}$, ζ_t^a, ζ_t^d, ζ_t^m.

This decomposition of risk premium is interesting because it is similar to the Autorgressive conditional heteroscedasticity in mean (ARCH - M) process in Engle, Lilien and Robin (1987). It implies that, the conditional volatilities have a direct effect on the conditional means of the variables. However, there are two differences between this model and the standard statistical ARCH - M model. First the coefficients in this model are restricted structural parameters and have economic meaning instead of free parameters. Second, the relevant conditional volatility in Engle, Lilien and Robin (1987) is that of the realised risk premium itself whereas in this model it is economic agents expectations about future shocks volatility. This is because agents are forward - looking in this model. Any change in expected future volatility has an immediate impact on current decision rules and asset prices. Consequently, this decomposition will allow us to investigate the link between risk premia and macroeconomic variables as well as structural parameters.

Andreasen (2011), Ruge - Murcia (2010) among others show that the coefficients $rp_{\sigma\sigma\sigma}^{\ell,j} = 0$, $j = a, d, m$ when the innovations of the shocks display symmetric distributions. Since we assume normal distribution for the innovations, these coefficients are then equal to zero. That means that, third order approximations will have a small effect on the size of the risk premium compared to second - order approximations. Moreover, when the variances of the shocks are constant over time, the risk premium expression in (49) reduces to the first two lines. The last two lines outline the contribution of time - varying volatility to the dynamics of risk premium.

Clearly the dynamics of the risk premium will be driven by the state as long as the coefficients $rp^{\ell}_{\sigma\sigma v} \neq 0$. Unlike in the second order approximation case, the price of risk is time-varying at third order approximation. The dynamics of the price of risk in this case is driven by the shock innovations and the state variables.

We now turn to examine how the three volatility shocks contribute to the level as well as the variations of risk premia at different maturities at the SMM parameter estimates. To that end, the coefficients $rp^{\ell}_{\sigma\sigma,j}$, $j = \sigma_d, \sigma_a, \sigma_m$ in (49) for different maturities ℓ are plotted along with the unconditional means and standard deviations of risk premia.

Figure 9 explores how these coefficients are related to the unconditional means and standard deviations of risk premia computed based on 150000 simulated observations. The horizontal lines of figure 9 are maturity and the vertical line the values of the specified variables. Panel A plots the coefficients associated with the conditional volatility of the technology shock $rp^{\ell}_{\sigma\sigma,\sigma_a}$, panel B the coefficients associated with the conditional volatility of the monetary policy shock, $rp^{\ell}_{\sigma\sigma,\sigma_m}$, panel C the coefficients associated with the conditional volatility of the preferences shock, $rp^{\ell}_{\sigma\sigma,\sigma_d}$, and panel D the unconditional means and standard deviations of risk premium. In panel D the blue dotted line (left scale) represents the unconditional means while the green line (right axis) represents the unconditional standard deviations. It is clear from panel B that monetary policy conditional volatility plays a limited role in the means as well as the average of risk premia as its scaling coefficient $rp^{\ell}_{\sigma\sigma,\sigma_m}$ is negligible and is of order 10^{-9}. Moreover, if there is any contribution of the conditional volatility of the policy shock, that would concern only very short terms risk premia.

Figure 9 suggests that unconditional means and standard deviations of risk premia are mainly driven by conditional volatilities of productivity and preferences shocks. The coefficients ($rp^{\ell}_{\sigma\sigma,\sigma_a}$) associated with the productivity conditional volatility are positive while those ($rp^{\ell}_{\sigma\sigma,\sigma_d}$) associated with the preferences conditional volatility are negative for all maturities. It implies that the conditional volatility of productivity shock contributes positively whereas the conditional volatility of preferences shock contributes negatively to the averages of risk premia. Notice that the coefficient $rp^{\ell}_{\sigma\sigma,\sigma_a}$ is increasing with maturity. On the other hand, the coefficient $rp^{\ell}_{\sigma\sigma,\sigma_d}$ is decreasing with maturity at the short end of the yield curve (from 6m to 2y) and increasing from 2y to 10y. This also suggests that differences in the means and standard deviations of risk premia across maturities are partly explained by productivity and preferences conditional volatility. Moreover, the conditional volatility of productivity contributions to the averages and standard deviations of risk premia are increasing with the maturity. With regard to the conditional volatility of preferences shock, the averages of risk premia are more negatively affected from 3m to 2y - maturities. The reverse is true for maturities greater than 2y. The contributions to risk premia standard deviations is increasing with maturity from 3m to 2y - maturities and decreasing after the 2y maturity.

Now, we explore how changes in some structural parameters affect the means and variances of risk premia. The parameters considered are the habit formation parameter (b), the capital adjustment cost parameter (κ), the Epstein - Zin parameter (φ).

Figure 10 plots the averages and standard deviations of risk premia for different maturities obtained by changing the considered parameters from their SMM estimates. Panel A shows the averages and panel B the standard deviations.

We change the habit formation parameter from the baseline value of 0.57 to a high level of 0.95 with others parameters set at their baseline values. Habit formation preferences are known to positively magnify the size of risk premium in endowment economy. However, this result can be mitigated in production economies. When the capital stock adjustment cost parameter is fixed at the baseline value ($\kappa = 3.57$) changes in b have a negligible impact on the means as well as the standard deviation of risk premia. This is consistent with the finding in chapter 1 that the habit formation parameter only has a significant effect on the level of the risk premia when the capital stock is fixed ($\kappa = +\infty$). This result is also true for the adjustment cost parameter. When the habit formation parameter is fixed at $b = 5.67$, the adjustment cost parameter has little effect on risk premia. As for the Epstein - Zin parameter φ remember, it is the key determinant of the risk aversion parameter. So, increases in the absolute value of φ is expected to have positive impact on risk premia. We change φ from -167 to -200. As a result, the risk premia increase for all maturities. The 10y bond risk premia increases by 147 basis points from 1.78% to 3,25%. The standard deviation slightly increases for all maturities.

8 Conclusion

This paper studies the term structure of nominal bonds interest rates and risk premia in a New-keynesian framework with recursive preferences and time - varying uncertainty. Time - varying uncertainty is introduced by assuming that technology, preferences and monetary policy shocks conditional volatilities follow stochastic volatility processes. The model is solved by perturbation method which involves taking third - order Taylor series expansions. Then, the parameters of the model are estimated by simulated method of moments (SMM). The analysis focuses on the effect of uncertainty shocks on the term structure of interest rates and risk premia.

Introducing time - varying uncertainty in the analysis of the term structure is important because changes in uncertainty or volatilities have an impact on economic agents consumption or portfolio decisions. Moreover, previous studies (Rudebusch and Swanson, 2010, Andreasen *et al*, 2013) have shown that recursive preferences are appropriate in analysing jointly asset prices and business cycles as opposed to the standard preferences in New keynesian DSGE literature. Previous work that make use of recursive preferences to analyse the term structure have focused on the impact of the level of the shocks on the term structure (Andreasen *et al*, 2013) or have employed time - varying volatility but using an endownment economy framework (Doh, 2010, van Bingsberg et al, 2010, Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez, 2010). Since endownment economy framework implies that the equilibrium consumption is exogenous, it is important to extend this analysis to a fully - fledged production economy in order to understand the impact of different sources of time - varying uncertainty on the term structure.

It is shown that the introduction of time - varying volatility has a first order effect and induces an addititional dynamics to interest rates and risk premia. In fact, the conditional volatilities affect the conditional means of the term structure and contribute to its fluctuations. It means that time - varying uncertainty affects agents decisions and asset prices. This is interesting as the derived risk premia decision rules micmic the ARCH - M process introduced by Engle, Lilien and Robins (1987). The main difference here is that the coefficients of our ARCH - M are functions of stuctural parameters.

At the SMM parameters estimates, the model generates statistics which are qualitatively in line with the term structure data counterparts. Results show that positive level of productivity shocks have downward shifting effects on the yield curve whereas positive monetary policy level shocks flatten the yield curve and preferences shocks affect positively the slope of the yield curve.

With regard to the volatility shocks, real uncertainty shocks (technology and preferences) play the most important role in the level and variations of risk premia relative to nominal uncertainty shocks (monetary policy). Techonology conditional variance contributes positively to the averages and variances risk premia whereas preferences shock conditional volatility contributes negatively to the averages of risk premia.

Early studies on the term structure literature in macro finance literature, that assume fixed capital stock, have found that higher habit formation strength leads to higher risk premia. In this exercise, I find that this result depends on wheter the capital stock is fixed or not. When the capital stock is fixed, a higher habit formation parameter significantly increases the risk premium. However when the capital stock is allowed to vary, increases in habit strength parameter leads to decreases in risk premiums. This is because allowing the capital stock to vary costlessly, open an additional channel for consumption smoothing.

Furthermore, the relative importance of the three shocks depends on the maturity of the bond. At the short-term of the bond yield curve (from three-period to five-period maturity), technology shock is relatively more important followed by the preference shock. From the middle to the end of the yield curve, preference shock dominates in terms of contribution as the technology shock relative importance is declining. Monetary policy shock is only more important than the preference shock for the two-period maturity. However, the relative dominances of the preference and technology shocks are explained by their persistence and size of standard deviations. When the standard deviation is controlled, the relative importance can be analyzed through the prices of risk involved in each shock. It turns out that, monetary policy shock price per unit is more important than those of preference and technology shocks.

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9 Appendix

Table 2: Unit Roots Test

	Test Statistic	
Variable	ADF	PP
Growth Rate of GDP	-6.435*	-9.242^*
Growth Rate of consumption	-4.369*	-8.95^*
Growth Rate of investment	-5.59*	-7.95^*
log of hours worked	-6.39*	-6.90^*
Rate of Inflation	-2.102	-3.148^{**}
Interest Rates Spread 10 year - 3 month	-3.854^*	-4.274^*
10 year risk premium	-4.854^*	-7.274^*

Note: **,* indicate significance at the 1%, 5% levels, respecively.

Table 3: baseline calibrated parameters

parameters	**description**	**value**
μ^z	long-run growth of productivity	1.0052
μ^Υ	long-run growth of investment shock	1.0016
n_{ss}	adjustment cost parameter	0.34
α	share of capital income	0.33
δ	depreciation rate	0.02
θ	elasticity of substitution among goods	11
θ_p	proportion of firms not adjusting price	0.75
π^{ss}	long-run gross inflation rate	1.008

Table 4: SMM Estimation

Description	Symbol	Time-varying volatility	Constant volatility
Preferences parameters			
Discount factor	β	0.9926	0.9928
		(0.0002)	(0.003)
Consumption curvature	γ	1.572	1.785
		(0.254)	(0.325)
EZ parameter	φ	-167.02	-199.346
		(25.02)	(30.52)
abour elasticity	ϕ	6.612	6.609
		(3.79)	(3.4)
Habit formation	b	0.570	077
		(0.0005)	(0.0002)
Capital adjust. cost parameter	κ	3.565	3.621
		(0.111)	(0.371)
Policy rule parameters			
AR parameter	ρ_i	0.663	0.663
		(0.0007)	(0.0027
Inflation reaction coeff	γ_π	3.225	3.224
		(1.005)	(1.255)
Output reaction coeff	γ_y	0.430	0.430
		(0.000)	(0.001)
Preferences shock parameters			
Persistence parameter	ρ_d	0.982	0.968
		(0.0052)	(0.031)
Standard deviation	$\bar{\sigma}_d$	0.014	0.014
		(0.002)	(0.001)
SV persistence	ρ_{σ_d}	0.605	-
		(0.236)	-
Standard deviation	σ_{σ_d}	0.400	-
		(0.023)	-
Productivity shock parameters			
Persistence parameter	ρ_a	0.948	0.960
		(0.0001)	(0.0023)
Standard deviation	$\bar{\sigma}_a$	0.012	0.008
		(0.000)	(0.0056)
SV persistence	ρ_{σ_a}	0.815	-
		(0.0641)	-
Standard deviation	σ_{σ_a}	0.420	-
		(0.000)	-
Monetary policy shock parameters			
Standard deviation	$\bar{\sigma}_m$	0.0015	1.47×10^{-5}
		(0.001)	(0.290)
SV persistence	ρ_{σ_m}	0.432	-
		(0.021)	-
Standard deviation	σ_{σ_m}	0.0053	-
		(0.0002))	-

Note: Asymptotic standard deviations in parenthesis

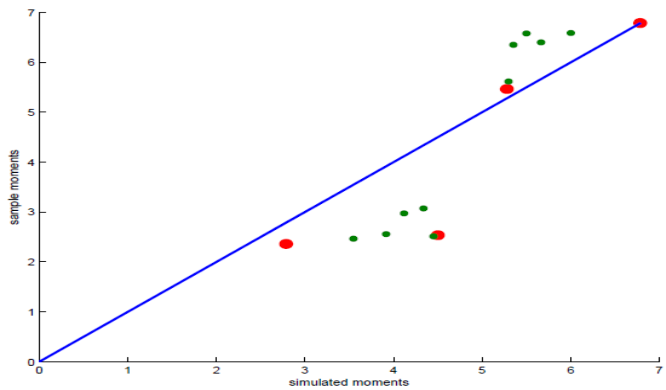
Table 5: Model Fit

	Data	SMM
Means		
$\Delta y_t \times 400$	2.183	2.215
$\Delta c_t \times 400$	2.010	2.217
$\Delta x_t \times 400$	2.316	3.035
$\pi_t \times 400$	4.303	3.245
$\log h_t \times 100$	-108.310	-107.830
$i_t^1 \times 400$	5.512	5.281
$i_t^{40} \times 400$	6.790	6.786
$rp_t^{40} \times 400$	1.781	1.695
Standard deviations		
$\Delta y_t \times 400$	3.322	4.153
$\Delta c_t \times 400$	2.236	2.527
$\Delta x_t \times 400$	8.648	9.14
$\pi_t \times 400$	2.943	4.204
$\log h_t \times 100$	1.520	2.052
$i_t^1 \times 400$	2.525	4.501
$i_t^{40} \times 400$	2.361	2.587
$rp_t^{40} \times 400$	23.704	14.27
First order autocorrelation		
$\Delta y_t \times 400$	0.260	0.127
$\Delta c_t \times 400$	0.437	0.502
$\Delta x_t \times 400$	0.495	0.380
$\pi_t \times 400$	0.816	0.933
$\log h_t \times 100$	0.922	0.960
$i_t^1 \times 400$	0.930	0.974
$i_t^{40} \times 400$	0.968	0.967
$rp_t^{40} \times 400$	-0.047	0.001
Second order autocorrelation		
$\Delta y_t \times 400$	0.224	0.014
$\Delta c_t \times 400$	0.233	0.239
$\Delta x_t \times 400$	0.405	0.051
$\pi_t \times 400$	0.737	0.900
$\log h_t \times 100$	0.822	0.936
$i_t^1 \times 400$	0.886	0.943
$i_t^{40} \times 400$	0.940	0.936
$rp_t^{40} \times 400$	0.066	0.245

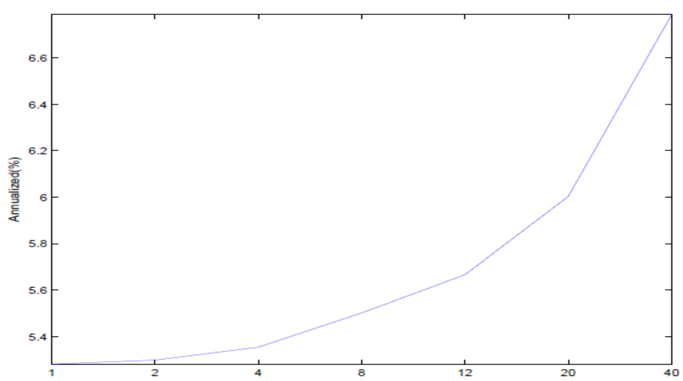
Note: Model moments are computed based on 50,000 simulated observations

Figure 1: Model Implied Term Structure of Interest Rates

Panel A: Model Fit



Panel B: Unconditional Means of Interest Rates



Panel C: Unconditional Standard Deviations of Interest Rates

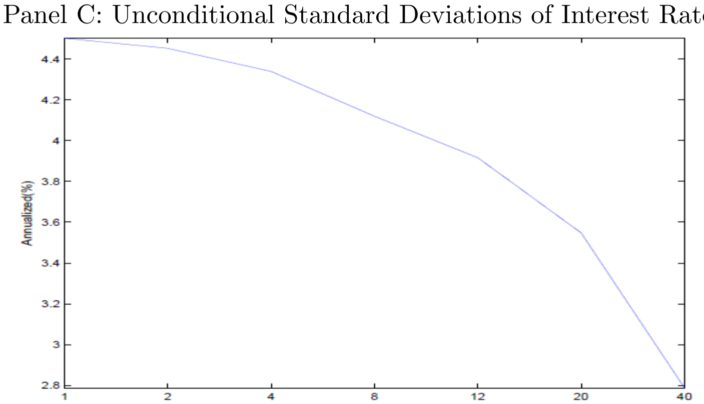


Figure 2: Simulated Series of the Term Structure

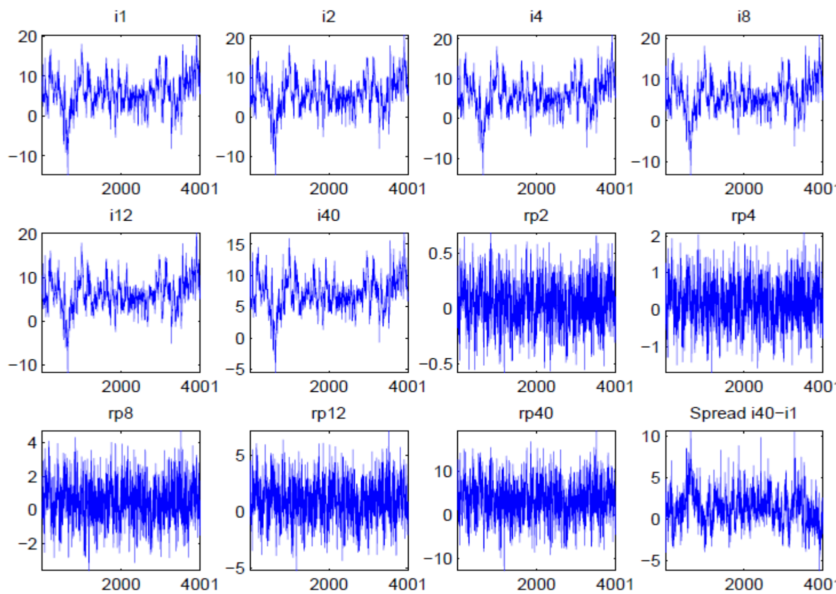


Figure 3: Responses to Productivity Level Shock

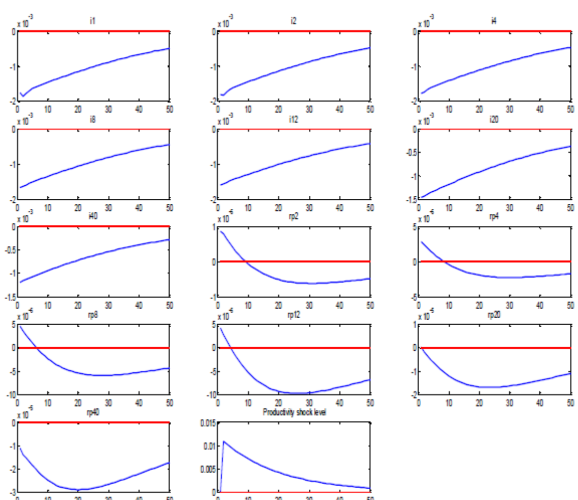


Figure 4: Responses to Preferences Level Shock

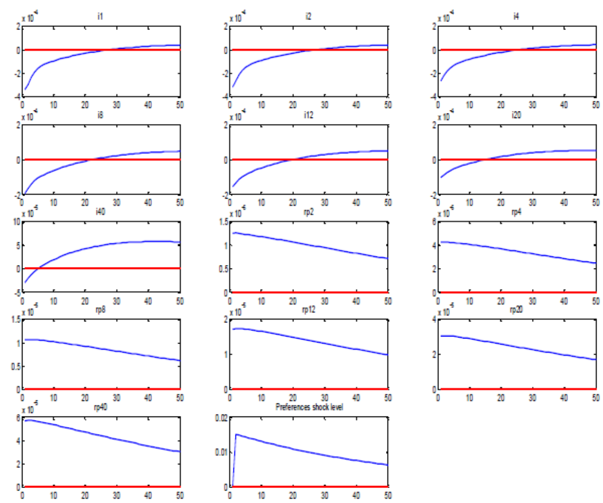


Figure 5: Responses to the Monetary Policy Level Shock

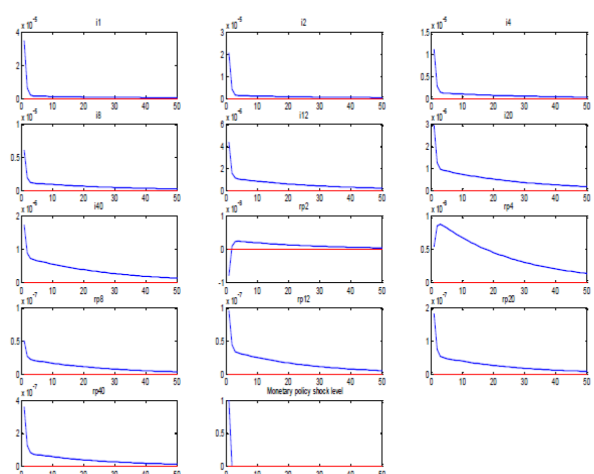


Figure 6: Responses to the Productivity Volatility Shock

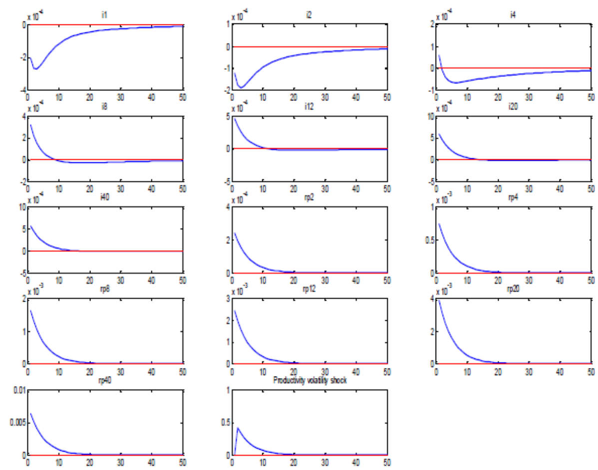


Figure 7: Responses to Preferences Volatility Shock

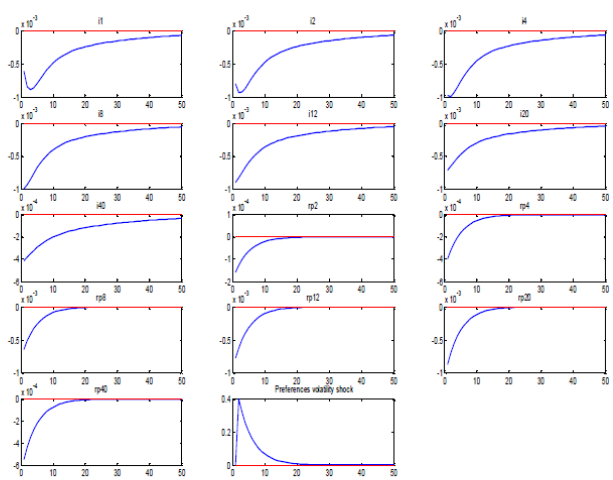


Figure 8: Responses to Policy Volatility Shock

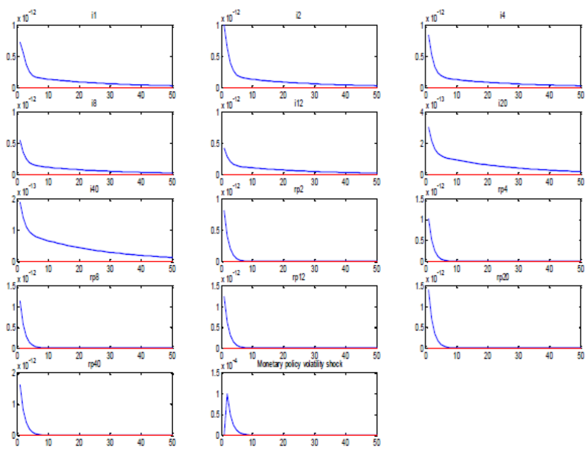
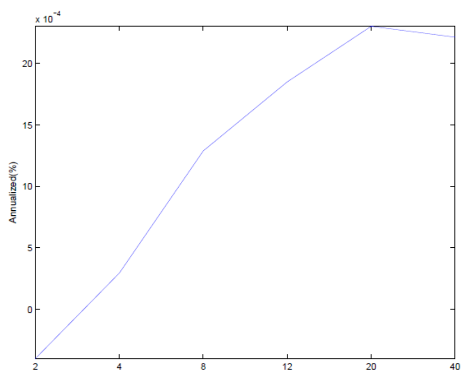
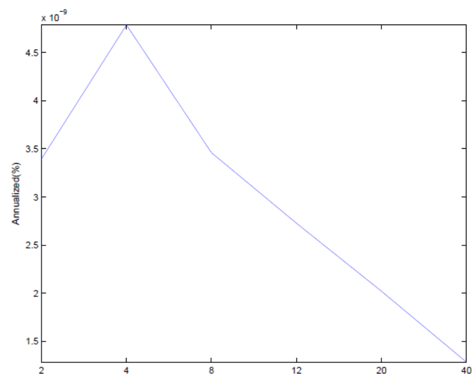


Figure 9: Risk Premium and Conditional Volatility Effects Coefficients

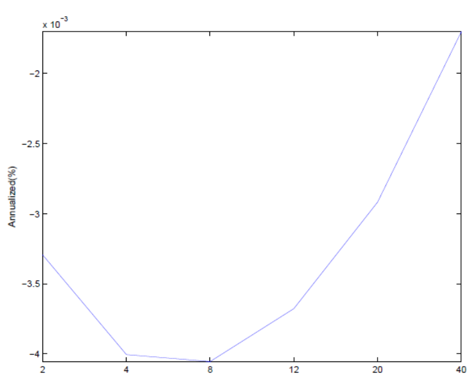
Panel A: Productivity Volatility Shock $rp^{\ell}_{\sigma\sigma,\sigma_a}$



Panel B: Monetary Policy Volatility Shock $rp^{\ell}_{\sigma\sigma,\sigma_m}$



Panel C: Preferences Volatility Shock $rp^{\ell}_{\sigma\sigma,\sigma_d}$



Panel D: Unconditional Means and Standard Deviations of Risk Premia